Probabilistic Termination by Monadic Affine Sized Typing

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Motivations

- Probabilistic programming languages are important in computer science: modeling uncertainty, cryptography, machine learning, AI...
- Quantitative notion of termination: almost-sure termination (AST)
- AST has been studied for imperative programs in the last years...
- ...but what about the functional probabilistic languages?

We introduce a monadic, affine sized type system sound for AST.

Sized types: the deterministic case

Simply-typed λ-calculus is strongly normalizing (SN).

No longer true with the letrec construction...

Sized types: a decidable extension of the simple type system ensuring SN for λ-terms with letrec.

See notably:

- Hughes-Pareto-Sabry 1996, *Proving the correctness of reactive systems using sized types*,
- Barthe-Frade-Giménez-Pinto-Uustalu 2004, *Type-based termination of recursive definitions*.

Sized types: the deterministic case

Sizes: $\mathfrak{s}, \mathfrak{r} \quad ::= \quad \mathfrak{i} \mid \infty \mid \widehat{\mathfrak{s}}$

+ size comparison inducing subtyping. Notably $\widehat{\infty} = \infty$.

Idea:

- $\mathsf{Nat}^{\widehat{\mathfrak{i}}}$ is 0,
- $\mathsf{Nat}^{\widehat{\widehat{\mathfrak{i}}}}$ is 0 or S 0,
- ...
- Nat^{∞} is any natural number. Often denoted simply Nat.

The same for lists,...

Sized types: the deterministic case

Sizes: $\mathfrak{s}, \mathfrak{r} \quad ::= \quad \mathfrak{i} \mid \infty \mid \widehat{\mathfrak{s}}$

+ size comparison inducing subtyping. Notably $\widehat{\infty} = \infty$.

Fixpoint rule:

$$\frac{\Gamma, f : \mathsf{Nat}^{\mathfrak{i}} \to \sigma \vdash M : \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \sigma[\mathfrak{i}/\widehat{\mathfrak{i}}] \qquad \mathfrak{i} \text{ pos } \sigma}{\Gamma \vdash \mathsf{letrec}\ f \ = \ M : \mathsf{Nat}^{\mathfrak{s}} \to \sigma[\mathfrak{i}/\mathfrak{s}]}$$

Typable $\implies$ SN. Proof using reducibility candidates.

Decidable type inference.

Sized types: example in the deterministic case

From Barthe et al. (op. cit.):

$$
\begin{array}{lll}
\text{plus} \equiv (\textsf{letrec} & \mathit{plus}_{:\text{Nat}^{\imath}\to\text{Nat}\to\text{Nat}} = & \\
& \lambda x_{:\text{Nat}^{\hat{\imath}}}.\ \lambda y_{:\text{Nat}}.\ \textsf{case}\ x\ \textsf{of}\ \{\textsf{o} \Rightarrow y & \\
& \qquad\qquad\qquad\qquad \mid \textsf{s} \Rightarrow \lambda x'_{:\text{Nat}^{\imath}}.\ \textsf{s}\ \underbrace{(\mathit{plus}\ x'\ y)}_{:\text{Nat}} & \\
& \qquad\qquad\qquad\qquad \} & \\
) : & \text{Nat}^{s} \to \text{Nat} \to \text{Nat} &
\end{array}
$$

Size decreases during recursive calls $\Rightarrow$ SN.

A probabilistic λ-calculus

$$\begin{array}{rcl} M, N, \ldots & ::= & V \mid V\ V \mid \text{let } x = M \text{ in } N \mid M \oplus_p N \\ & & \mid \text{case } V \text{ of } \{ \mathsf{S} \to W \mid 0 \to Z \} \end{array}$$

$$V, W, Z, \ldots \quad ::= \quad x \mid 0 \mid \mathsf{S}\ V \mid \lambda x.M \mid \text{letrec } f = V$$

- Formulation equivalent to λ-calculus with $\oplus_p$, but constrained for technical reasons (Let normal form)
- Restriction to base type Nat for simplicity, but can be extended to general inductive datatypes (as in sized types)

A probabilistic λ-calculus: operational semantics

$$\frac{}{\text{let } x = V \text{ in } M \ \to_v \ \left\{ (M[x/V])^1 \right\}}$$

$$\frac{}{(\lambda x.M)\ V \ \to_v \ \left\{ (M[x/V])^1 \right\}}$$

$$\frac{}{(\text{letrec } f = V)\ \left(c\ \overrightarrow{W}\right) \ \to_v \ \left\{ \left(V[f/(\text{letrec } f = V)]\ \left(c\ \overrightarrow{W}\right)\right)^1 \right\}}$$

A probabilistic λ-calculus: operational semantics

$$\frac{}{\mathsf{case}\ \mathsf{S}\ V\ \mathsf{of}\ \{\,\mathsf{S} \rightarrow W \ \mid\ 0 \rightarrow Z\,\} \ \rightarrow_v\ \left\{ (W\ \ V)^1 \right\}}$$

$$\frac{}{\mathsf{case}\ 0\ \mathsf{of}\ \{\,\mathsf{S} \rightarrow W \ \mid\ 0 \rightarrow Z\,\} \ \rightarrow_v\ \left\{ (Z)^1 \right\}}$$

A probabilistic λ-calculus: operational semantics

$$\frac{}{M \oplus_p N \;\rightarrow_v\; \{\, M^p,\, N^{1-p} \,\}}$$

$$\frac{M \;\rightarrow_v\; \{\, L_i^{p_i} \mid i \in I \,\}}{\text{let } x = M \text{ in } N \;\rightarrow_v\; \{\, (\text{let } x = L_i \text{ in } N)^{p_i} \mid i \in I \,\}}$$

A probabilistic λ-calculus: operational semantics

$$\frac{\mathscr{D} \stackrel{VD}{=} \left\{ M_j^{p_j} \;\middle|\; j \in J \right\} + \mathscr{D}_V \qquad \forall j \in J,\quad M_j \;\rightarrow_v\; \mathscr{E}_j}{\mathscr{D} \;\rightarrow_v\; \left(\sum_{j \in J} p_j \cdot \mathscr{E}_j\right) + \mathscr{D}_V}$$

For $\mathscr{D}$ a distribution of terms:

$$[\![\, \mathscr{D} \,]\!] \;=\; \sup_{n \in \mathbb{N}} \left(\left\{ \mathscr{D}_n \;\middle|\; \mathscr{D} \Rrightarrow_v^n \mathscr{D}_n \right\} \right)$$

where $\Rrightarrow_v^n$ is $\rightarrow_v^n$ followed by projection on values.

We let $[\![\, M \,]\!] \;=\; [\![\, \{ \, M^1 \, \} \,]\!]$.

M is AST iff $\sum [\![\, M \,]\!] \;=\; 1$.

Random walks as probabilistic terms

- Biased random walk:

$$M_{bias} \;=\; \Big(\text{letrec } f \;=\; \lambda x.\text{case } x \text{ of } \Big\{ \mathsf{S} \rightarrow \lambda y.f(y) \oplus_{\frac{2}{3}} (f(\mathsf{S}\,\mathsf{S}\,y))) \;\mid\; 0 \rightarrow 0 \Big\}\Big)\; \underline{n}$$

- Unbiased random walk:

$$M_{unb} \;=\; \Big(\text{letrec } f \;=\; \lambda x.\text{case } x \text{ of } \Big\{ \mathsf{S} \rightarrow \lambda y.f(y) \oplus_{\frac{1}{2}} (f(\mathsf{S}\,\mathsf{S}\,y))) \;\mid\; 0 \rightarrow 0 \Big\}\Big)\; \underline{n}$$

$$\sum [\![M_{bias}]\!] \;=\; \sum [\![M_{unb}]\!] \;=\; 1$$

Capture this in a sized type system?

Another term

We also want to capture terms as:

$$M_{nat} \;=\; \Big(\mathsf{letrec}\ f \;=\; \lambda x.x \oplus_{\frac{1}{2}} \mathsf{S}\ (f\ x)\Big)\ 0$$

of semantics

$$[\![\, M_{nat} \,]\!] \;=\; \Big\{(0)^{\frac{1}{2}}, (\mathsf{S}\ 0)^{\frac{1}{4}}, (\mathsf{S}\ \mathsf{S}\ 0)^{\frac{1}{8}}, \ldots\Big\}$$

summing to 1.

Beyond SN terms, towards distribution types

First idea: extend the sized type system with:

$$\text{Choice} \quad \frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M \oplus_p N : \sigma}$$

and "unify" types of M and N by subtyping.

Kind of product interpretation of $\oplus$: we can't capture more than SN. . .

Beyond SN terms, towards distribution types

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and "unify" types of M and N by subtyping.

We can't type M_{bias} nor M_{unb} in a way decreasing the size (essential for letrec): we get at best

$$f \ : \ \mathsf{Nat}^{\widehat{\widehat{\imath}}} \to \mathsf{Nat}^{\infty} \ \vdash \ \lambda y. f(y) \oplus_{\frac{2}{3}} (f(\mathsf{S}\,\mathsf{S}\,y))) \ : \ \mathsf{Nat}^{\widehat{\imath}} \to \mathsf{Nat}^{\infty}$$

and can't use a variation of the letrec rule on that.

Beyond SN terms, towards distribution types

We will use distribution types, built as follows:

$$\text{Choice} \quad \frac{\Gamma \mid \Theta \vdash M : \mu \qquad \Gamma \mid \Psi \vdash N : \nu \qquad \{\!|\, \mu \,|\!\} = \{\!|\, \nu \,|\!\}}{\Gamma \mid \Theta \oplus_p \Psi \vdash M \oplus_p N : \mu \oplus_p \nu}$$

Now

$$f : \left\{ \left(\mathsf{Nat}^{\mathfrak{i}} \to \mathsf{Nat}^{\infty}\right)^{\frac{2}{3}}, \left(\mathsf{Nat}^{\widehat{\widehat{\mathfrak{i}}}} \to \mathsf{Nat}^{\infty}\right)^{\frac{1}{3}} \right\}$$

$$\vdash$$

$$\lambda y.f(y) \oplus_{\frac{2}{3}} (f(\mathsf{S}\,\mathsf{S}\,y))) \quad : \quad \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \mathsf{Nat}^{\infty}$$

Beyond SN terms, towards distribution types

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Similarly:

$$f : \left\{ \left(\mathsf{Nat}^{\mathfrak{i}} \to \mathsf{Nat}^{\infty}\right)^{\frac{1}{2}},\ \left(\mathsf{Nat}^{\widehat{\widehat{\mathfrak{i}}}} \to \mathsf{Nat}^{\infty}\right)^{\frac{1}{2}} \right\}$$

$$\vdash$$

$$\lambda y. f(y) \oplus_{\frac{1}{2}} (f(\mathsf{S}\,\mathsf{S}\,y))) \ :\ \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \mathsf{Nat}^{\infty}$$

Designing the fixpoint rule

$$\{\!|\, \Gamma \,|\!\} \;=\; \mathsf{Nat}$$

$$\mathfrak{i} \notin \Gamma \text{ and } \mathfrak{i} \text{ positive in } \nu$$

$$??$$

$$\text{LetRec}\quad \frac{\Gamma \,|\, f \,:\, \left\{ (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \;\middle|\; j \in J \right\} \vdash V \,:\, \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \nu[\widehat{\mathfrak{i}}/\mathfrak{i}]}{\Gamma \,|\, \emptyset \vdash \mathsf{letrec}\ f \;=\; V \,:\, \mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{r}/\mathfrak{i}]}$$

Designing the fixpoint rule

$$\text{LetRec}\quad \frac{\{\!|\,\Gamma\,|\!\} = \mathsf{Nat} \qquad \mathfrak{i} \notin \Gamma \text{ and } \mathfrak{i} \text{ positive in } \nu \qquad {\color{red}\sum_{j \in J}\ p_j (\!|\mathfrak{s}_j|\!) < 1} \qquad \Gamma \mid f : \left\{ (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \ \middle|\ j \in J \right\} \vdash V : \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \nu[\widehat{\mathfrak{i}}/\mathfrak{i}]}{\Gamma \mid \emptyset \vdash \mathsf{letrec}\ f = V : \mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{r}/\mathfrak{i}]}$$

would allow to type

$$M_{bias} = \left(\mathsf{letrec}\ f = \lambda x.\mathsf{case}\ x\ \mathsf{of}\ \left\{ \mathsf{S} \to \lambda y.f(y) \oplus_{\frac{2}{3}} (f(\mathsf{S}\,\mathsf{S}\,y))) \ \middle|\ 0 \to 0 \right\}\right)\ \underline{n}$$

Designing the fixpoint rule

$$\frac{\begin{array}{c}\{\!|\, \Gamma \,|\!\} \;=\; \mathsf{Nat} \\ \mathfrak{i} \notin \Gamma \text{ and } \mathfrak{i} \text{ positive in } \nu \\ \color{red}{\sum_{j \in J}\; p_j (\!| \mathfrak{s}_j |\!) < 1 \text{ or } \sum_{j \in J}\; p_j < 1} \\ \Gamma \,|\, f \;:\; \left\{\, (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \;\;\middle|\;\; j \in J \,\right\} \vdash V \;:\; \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \nu[\widehat{\mathfrak{i}}/\mathfrak{i}]\end{array}}{\Gamma \,|\, \emptyset \vdash \mathsf{letrec}\ f \;=\; V \;:\; \mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{r}/\mathfrak{i}]}\ \mathsf{LetRec}$$

would allow to type M_{nat} too:

$$M_{nat} \;=\; \Big(\mathsf{letrec}\ f \;=\; \lambda x.x \oplus_{\frac{1}{2}} \mathsf{S}\ (f\ x)\Big)\ 0$$

Designing the fixpoint rule

$$\{\!|\, \Gamma \,|\!\} = \mathsf{Nat}$$

$$\mathfrak{i} \notin \Gamma \text{ and } \mathfrak{i} \text{ positive in } \nu$$

$$\sum_{j\in J} p_j (\!| \mathfrak{s}_j |\!) < 1 \text{ or } \sum_{j\in J} p_j < 1$$

$$\text{LetRec} \quad \frac{\Gamma \,|\, f : \left\{ (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \;\middle|\; j \in J \right\} \vdash V : \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \nu[\widehat{\mathfrak{i}}/\mathfrak{i}]}{\Gamma \,|\, \emptyset \vdash \mathsf{letrec}\ f = V : \mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{r}/\mathfrak{i}]}$$

But how to cope with

$$M_{unb} = \left(\mathsf{letrec}\ f = \lambda x.\mathsf{case}\ x\ \mathsf{of}\ \left\{ \mathsf{S} \to \lambda y.f(y) \oplus_{\frac{1}{2}} (f(\mathsf{S}\,\mathsf{S}\,y))) \;\middle|\; 0 \to 0 \right\} \right) \underline{n}$$

Designing the fixpoint rule

$$\text{LetRec}\quad \frac{\begin{array}{c}\{\!|\,\Gamma\,|\!\} \;=\; \mathsf{Nat} \\ \mathfrak{i} \notin \Gamma \text{ and } \mathfrak{i} \text{ positive in } \nu \\ \color{red}{\left\{\, (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \;\;\middle|\;\; j \in J \,\right\} \text{ induces an AST sized walk}} \\ \Gamma \,|\, f \;:\; \left\{\, (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{s}_j/\mathfrak{i}])^{p_j} \;\;\middle|\;\; j \in J \,\right\} \vdash V \;:\; \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \nu[\widehat{\mathfrak{i}}/\mathfrak{i}]\end{array}}{\Gamma \,|\, \emptyset \vdash \mathsf{letrec}\ f \;=\; V \;:\; \mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{r}/\mathfrak{i}]}$$

solves the problem for

$$M_{unb} \;=\; \left(\mathsf{letrec}\ f \;=\; \lambda x.\mathsf{case}\ x\ \mathsf{of}\ \left\{ \mathsf{S} \to \lambda y.f(y) \oplus_{\frac{1}{2}} (f(\mathsf{S}\,\mathsf{S}\,y))) \;\;\middle|\;\; 0 \to 0 \right\}\right)\ \underline{n}$$

by deferring to an external PTIME procedure the convergence checking.

Generalized random walks and the necessity of affinity

A crucial feature: our type system is affine. Higher-order symbols occur at most once. Why? Consider:

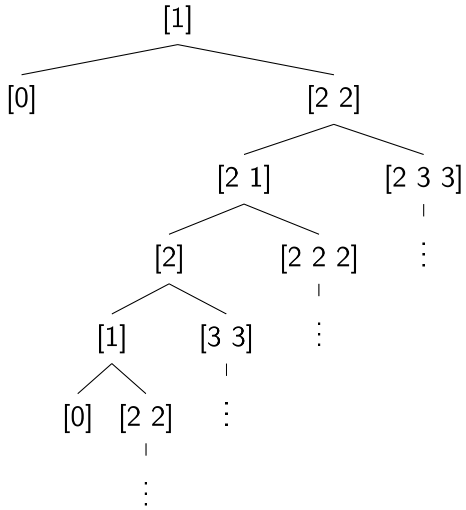
$$M_{naff} \;=\; \text{letrec } f \;=\; \lambda x.\text{case } x \text{ of } \left\{ \mathsf{S} \to \lambda y.f(y) \oplus_{\frac{2}{3}} (f(\mathsf{S}\,\mathsf{S}\,y)\,;\, f(\mathsf{S}\,\mathsf{S}\,y)) \;\middle|\; 0 \to 0 \right\}$$

and recall that its affine version was AST. Some reductions:

$$\begin{array}{rcl}
M_{naff}\ (\mathsf{S}\,0) & \to_v^* & 0 \\
M_{naff}\ (\mathsf{S}\,0) & \to_v^* & M_{naff}\ (\mathsf{S}\,\mathsf{S}\,0)\,;\ M_{naff}\ (\mathsf{S}\,\mathsf{S}\,0) \\
 & \to_v^* & M_{naff}\ (\mathsf{S}\,0)\,;\ M_{naff}\ (\mathsf{S}\,\mathsf{S}\,0) \\
 & \to_v^* & M_{naff}\ 0\,;\ M_{naff}\ (\mathsf{S}\,\mathsf{S}\,0) \\
 & \to_v^* & 0\,;\ M_{naff}\ (\mathsf{S}\,\mathsf{S}\,0) \\
 & \to_v^* & M_{naff}\ (\mathsf{S}\,\mathsf{S}\,0) \\
M_{naff}\ (\mathsf{S}\,0) & \to_v^* & M_{naff}\ (\mathsf{S}\,\mathsf{S}\,\mathsf{S}\,0)\,;\ M_{naff}\ (\mathsf{S}\,\mathsf{S}\,\mathsf{S}\,0)\,;\ M_{naff}\ (\mathsf{S}\,\mathsf{S}\,0) \\
 & \to_v^* & 0
\end{array}$$

Generalized random walks and the necessity of affinity

Tree of recursive calls:



Leftmost edges have probability $\frac{2}{3}$; rightmost ones $\frac{1}{3}$.

Generalized random walks and the necessity of affinity

Local shape

$$[i_1 \; \cdots \; i_k]$$

$$[i_1 \; \cdots \; i_k - 1] \quad [i_1 \; \cdots \; i_k + 1 \; i_k + 1]$$

when $i_k > 1$, and

$$[i_1 \; \cdots \; i_{k-1} \; 1]$$

$$[i_1 \; \cdots \; i_{k-1}] \quad [i_1 \; \cdots \; i_{k-1} \; 2 \; 2]$$

else. Leaves are all labeled with [0].

The rightmost branch always increases the sum $i_1 + \cdots + i_k$ by at least 3 $\rightarrow$ non AST random walk.

Affinity ensures that modeling recursion with an “usual” random walk (and not one on stacks) is sound.

Affinity and implicit complexity

Affinity for probabilistic λ-calculus has been used to capture probabilistic functions computable in PTIME. See:

- Cappai-dal Lago 2015, *On Equivalences, Metrics, and Polynomial Time*
- dal Lago-Parisen Toldin 2015, *A higher-order characterization of probabilistic polynomial time*

Decoration of base type Str with two modalities:

- $\blacksquare A \to B$: constant time computation
- $\square A \to B$: PTIME computation

Affinity and implicit complexity

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Differences here:

- Focus on termination vs. PTIME, and thus different type annotations
- Allow non-terminating probabilistic branches vs. all branches computable in PTIME

Key points

- Affine type system
- Distribution types
- Sized walks induced by the letrec rule and solved by an external PTIME procedure

Key property I: subject reduction

Main idea: reduction of

$$\emptyset \mid \emptyset \vdash 0 \oplus 0 \;:\; \left\{ \left(\mathsf{Nat}^{\widehat{\mathfrak{s}}}\right)^{\frac{1}{2}}, \left(\mathsf{Nat}^{\widehat{\widehat{\mathfrak{r}}}}\right)^{\frac{1}{2}} \right\}$$

is to

$$\left\{ \left(0 \;:\; \mathsf{Nat}^{\widehat{\mathfrak{s}}}\right)^{\frac{1}{2}}, \left(0 \;:\; \mathsf{Nat}^{\widehat{\widehat{\mathfrak{r}}}}\right)^{\frac{1}{2}} \right\}$$

1. Same expectation type
2. Splitting of $[\![0 \oplus 0]\!]$ in a typed representation $\rightarrow$ notion of pseudo-representation

Key property I: subject reduction

Lemma (Subject reduction)

Let $M \in \Lambda_{\oplus}$ *be a closed term and* $\mathscr{D} = \left\{ N_i^{p_i} \mid i \in I \right\}$ *be the unique closed term distribution such that* $M \to_v \mathscr{D}$.

Suppose that $\emptyset \,|\, \emptyset \vdash M : \mu$*, then there exists a closed typed distribution* $\left\{ (L_j : \nu_j)^{p'_j} \mid j \in J \right\}$ *such that*

- $\mathbb{E}\left((L_j : \nu_j)^{p'_j} \right) = \mu,$
- $\left[(L_j)^{p'_j} \mid j \in J \right]$ *is a pseudo-representation of* $\left\{ (N_i)^{p_i} \mid i \in I \right\}$.

Key property I: subject reduction

Theorem

Let $M \in \Lambda_\oplus$ *be such that* $\emptyset \mid \emptyset \vdash M : \mu$*. Then there exists a closed typed distribution* $\left\{ (W_j : \sigma_j)^{p'_j} \mid j \in J \right\}$ *such that*

- $\mathbb{E}\left((W_j : \sigma_j)^{p'_j} \right) \preccurlyeq \mu,$
- *and that* $\left[(W_j)^{p'_j} \mid j \in J \right]$ *is a pseudo-representation of* $[\![M]\!]$.

Key properties

Theorem (Typing soundness)

If $\Gamma \mid \Theta \vdash M : \mu$, then M is AST.

Proof by reducibility, using set of candidates parametrized by probabilities.

Sets of candidates

$$\mathsf{TRedFin}^{p}_{\mu,\rho} \;=\; \left\{ M \in \Lambda_{\oplus}(\{\!|\,\mu\,|\!\}) \;\middle|\; \begin{array}{r} \forall 0 \leq r < p,\;\; \exists \nu_r \preccurlyeq \mu,\;\; \exists n_r \in \mathbb{N}, \\ M \Rrightarrow_{v}^{n_r} \mathscr{D}_r \text{ and } \mathscr{D}_r \in \mathsf{DRed}^{r}_{\nu_r,\rho} \end{array} \right\}$$

To compare with:

Theorem

Let $M \in \Lambda_{\oplus}$ be such that $\emptyset \,|\, \emptyset \vdash M : \mu$. Then there exists a closed typed distribution $\left\{ (W_j : \sigma_j)^{p'_j} \;\middle|\; j \in J \right\}$ such that

- $\mathbb{E}\left((W_j : \sigma_j)^{p'_j} \right) \;\preccurlyeq\; \mu$,
- *and that* $\left[(W_j)^{p'_j} \;\middle|\; j \in J \right]$ *is a pseudo-representation of* $[\![\, M \,]\!]$.

Sets of candidates

$\mathrm{DRed}^{p}_{\mu,\rho}$ is the set of finite distributions of values (in the sense that they have a finite support) admitting a pseudo-representation $\mathscr{D} = \left[(V_i)^{p_i} \;\middle|\; i \in I \right]$ such that, setting $\mu = \left\{ (\sigma_j)^{p'_j} \;\middle|\; j \in J \right\}$, there exists families $(p_{ij})_{i\in I, j\in J}$ and $(q_{ij})_{i\in I, j\in J}$ of reals of $[0,1]$ satisfying:

1. $\forall i \in I, \quad \forall j \in J, \quad V_i \in \mathrm{VRed}^{q_{ij}}_{\sigma_j,\rho}$,
2. $\forall i \in I, \quad \sum_{j\in J} p_{ij} = p_i$,
3. $\forall j \in J, \quad \sum_{i\in I} p_{ij} = \mu(\sigma_j)$,
4. $p \leq \sum_{i\in I} \sum_{j\in J} q_{ij} p_{ij}$.

μ is assumed to be uniform: $\forall j_1, j_2 \in J, \quad \{\!|\, \sigma_{j_1} \,|\!\} = \{\!|\, \sigma_{j_2} \,|\!\}$. Note also that (2) and (3) imply that $\sum \mathscr{D} = \sum \mu$.

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1. $\forall i \in I, \quad \forall j \in J, \quad V_i \in \mathrm{VRed}^{q_{ij}}_{\sigma_j,\rho},$
2. $\forall i \in I, \quad \sum_{j \in J} \; p_{ij} \; = \; p_i,$
3. $\forall j \in J, \quad \sum_{i \in I} \; p_{ij} \; = \; \mu(\sigma_j),$
4. $p \leq \sum_{i \in I} \sum_{j \in J} \; q_{ij} p_{ij}.$

μ is assumed to be uniform: $\forall j_1, j_2 \in J, \quad \{\!|\, \sigma_{j_1} \,|\!\} = \{\!|\, \sigma_{j_2} \,|\!\}$. Note also that (2) and (3) imply that $\sum \mathscr{D} = \sum \mu$.

Sets of candidates

$$\mathsf{VRed}^{p}_{\mathsf{Nat}^{\mathfrak{s}},\rho} \quad = \quad \left\{ \mathsf{S}^n 0 \;\middle|\; p > 0 \Longrightarrow n < [\![\mathfrak{s}]\!]_\rho \right\}$$

$$\begin{aligned}\mathsf{VRed}^{p}_{\sigma\to\mu,\rho} \quad = \quad \Big\{ V \in \Lambda^{V}_{\oplus}(\{\!|\, \sigma \to \mu \,|\!\}) \;\Big|\; & \forall q \in (0,1],\; \forall W \in \mathsf{VRed}^{q}_{\sigma,\rho}, \\ & V\ W \in \mathsf{TRedFin}^{pq}_{\mu,\rho} \Big\}\end{aligned}$$

Sets of candidates

Lemma

- *Let* $\mathscr{D} \in \mathrm{DRed}^{p}_{\mu,\rho}$. *Then* $\sum \mathscr{D} \geq p$.
- *Let* $M \in \mathrm{TRedFin}^{p}_{\mu,\rho}$. *Then* $\sum [\![M]\!] \geq p$.

So $M \in \mathrm{TRedFin}^{1}_{\mu,\rho} \;\Rightarrow\; M$ is AST.

Utility of the parameter p: for LetRec, computed as a limit.

Sets of candidates

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If $M \;=\; \text{letrec } f \;=\; V$,

1. For every ε, the sized walk argument ensures that after k_ε iterations we reach 0 with probability $\geq 1 - \varepsilon$.
2. We unfold the LetRec k_ε times and thus obtain a term in $\text{TRedFin}_{\mu,\rho}^{1-\varepsilon}$.
3. We prove that this implies that the folded term $M \in \text{TRedFin}_{\mu,\rho}^{1-\varepsilon}$.
4. Continuity on the TRedFin:

$$M \in \bigcap_{\varepsilon \in (0,1]} \text{TRedFin}_{\mu,\rho}^{1-\varepsilon} \quad \Longrightarrow \quad M \in \text{TRedFin}_{\mu,\rho}^{1} \quad \Longrightarrow \quad M \text{ AST}.$$

Sets of candidates

Extension to open terms:

$$\mathrm{OTRedFin}_{\mu,\rho}^{\Gamma \mid y : \{\tau_j^{p_j}\}_{j \in J}} = \Big\{ M \;\Big|\; \forall (q_i)_i \in [0,1]^n,\ \forall (V_1, \ldots, V_n) \in \textstyle\prod_{i=1}^n \mathrm{VRed}_{\sigma_i,\rho}^{q_i},\ \forall \left(q_j'\right)_j \in [0,1]^J,\ \forall W \in \bigcap_{j \in J} \mathrm{VRed}_{\sigma_j,\rho}^{q_j'},\ M[\overrightarrow{x}, y/\overrightarrow{V}, W] \in \mathrm{TRedFin}_{\mu,\rho}^{\left(\prod_{i=1}^n q_i\right)\left(\sum_{j \in J} p_j q_j'\right)} \Big\}$$

The reducibility main result:

Proposition

If $\Gamma \mid \Theta \vdash M : \mu$, *then* $M \in \mathrm{OTRedFin}_{\mu,\rho}^{\Gamma \mid \Theta}$ *for every* ρ.

implies typing soundness.

Conclusion

Main features of the type system:

- Affine type system
- Distribution types
- Sized walks induced by the letrec rule and solved by an external PTIME procedure

Subject reduction result on distributions.

Reducibility argument, parametrized by probabilities.

Next step: look for the type inference (decidable again??)

Thank you for your attention!

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Thank you for your attention!