Two Type-Theoretic Approaches to Probabilistic Termination

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Motivations

- Probabilistic programming languages are more and more pervasive in computer science: modeling uncertainty, robotics, cryptography, machine learning, AI...
- Quantitative notion of termination: almost-sure termination (AST)
- AST has been studied for imperative programs in the last years...
- ...but what about the probabilistic functional languages?

We introduce a monadic, affine sized type system sound for AST (our result at ESOP 2017), and sketch a dependent, affine type system for AST (work in progress).

Sized Types and Termination

A sound termination check for the deterministic case

Sized Types: the Deterministic Case

Simply-typed λ-calculus is strongly normalizing (SN).

No longer true with the letrec construction...

Sized types: a decidable extension of the simple type system ensuring SN for λ-terms with letrec.

See notably:

- Hughes-Pareto-Sabry 1996, *Proving the correctness of reactive systems using sized types*,
- Barthe-Frade-Giménez-Pinto-Uustalu 2004, *Type-based termination of recursive definitions*.

Sized Types: the Deterministic Case

Sizes: $\qquad \mathfrak{s}, \mathfrak{r} \quad ::= \quad \mathfrak{i} \mid \infty \mid \widehat{\mathfrak{s}}$

+ size comparison underlying subtyping. Notably $\widehat{\infty} \equiv \infty$.

Idea: k successors = at most k constructors.

- $\mathsf{Nat}^{\widehat{\mathfrak{i}}}$ is 0,
- $\mathsf{Nat}^{\widehat{\widehat{\mathfrak{i}}}}$ is 0 or S 0,
- ...
- Nat^{∞} is any natural number. Often denoted simply Nat.

The same for lists,...

Sized Types: the Deterministic Case

Sizes: $\mathfrak{s}, \mathfrak{r} \quad ::= \quad \mathfrak{i} \mid \infty \mid \widehat{\mathfrak{s}}$

$+$ size comparison underlying subtyping. Notably $\widehat{\infty} \equiv \infty$.

Fixpoint rule:

$$\frac{\Gamma, f \,:\, \mathsf{Nat}^{\mathfrak{i}} \to \sigma \vdash M \,:\, \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \sigma[\mathfrak{i}/\widehat{\mathfrak{i}}] \qquad \mathfrak{i} \text{ pos } \sigma}{\Gamma \vdash \mathsf{letrec}\ f \ = \ M \,:\, \mathsf{Nat}^{\mathfrak{s}} \to \sigma[\mathfrak{i}/\mathfrak{s}]}$$

"To define the action of f on size $n+1$,
we only call recursively f on size at most n"

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Typable $\implies$ SN. Proof using reducibility candidates.

Decidable type inference.

Sized types: example in the deterministic case

From Barthe et al. (op. cit.):

$$
\begin{array}{lll}
\text{plus} \equiv (\mathsf{letrec} & plus_{:\mathrm{Nat}^{\iota}\to\mathrm{Nat}\to\mathrm{Nat}} = & \\
& \lambda x_{:\mathrm{Nat}^{\hat{\iota}}}.\ \lambda y_{:\mathrm{Nat}}.\ \mathsf{case}\ x\ \mathsf{of}\ \{\mathsf{o} \Rightarrow y & \\
& & \mid \mathsf{s} \Rightarrow \lambda x'_{:\mathrm{Nat}^{\iota}}.\ \mathsf{s}\ \underbrace{(plus\ \ x'\ \ y)}_{:\mathrm{Nat}} \\
& & \} \\
)\ : & \mathrm{Nat}^{s} \to \mathrm{Nat} \to \mathrm{Nat} &
\end{array}
$$

The case rule ensures that the size of x' is lesser than the one of x.
Size decreases during recursive calls $\Rightarrow$ SN.

A Probabilistic Lambda-Calculus and its Operational Semantics

A Probabilistic λ-calculus

$$
\begin{array}{rcl}
M, N, \ldots & ::= & V \;\mid\; V\;V \;\mid\; \mathsf{let}\ x\ =\ M\ \mathsf{in}\ N \;\mid\; M \oplus_p N \\
& & \mid\; \mathsf{case}\ V\ \mathsf{of}\ \{\, \mathsf{S} \to W \;\mid\; 0 \to Z \,\}
\end{array}
$$

$$
V, W, Z, \ldots \quad ::= \quad x \;\mid\; 0 \;\mid\; \mathsf{S}\ V \;\mid\; \lambda x.M \;\mid\; \mathsf{letrec}\ f\ =\ V
$$

- Formulation equivalent to λ-calculus with $\oplus_p$, but constrained for technical reasons (A-normal form)
- Restriction to base type Nat for simplicity, but can be extended to general inductive datatypes (as in sized types)

A Probabilistic λ-calculus: Operational Semantics

$$\frac{}{\mathsf{let}\ x\ =\ V\ \mathsf{in}\ M\ \ \to_v\ \ \left\{(M[x/V])^1\right\}}$$

$$\frac{}{(\lambda x.M)\ \ V\ \ \to_v\ \ \left\{(M[x/V])^1\right\}}$$

$$\frac{}{(\mathsf{letrec}\ f\ =\ V)\ \left(c\ \overrightarrow{W}\right)\ \ \to_v\ \ \left\{\left(V[f/(\mathsf{letrec}\ f\ =\ V)]\ \left(c\ \overrightarrow{W}\right)\right)^1\right\}}$$

A Probabilistic λ-calculus: Operational Semantics

$$\frac{}{\mathsf{case}\ \mathsf{S}\ V\ \mathsf{of}\ \{\mathsf{S} \to W \mid 0 \to Z\} \to_v \left\{(W\ V)^1\right\}}$$

$$\frac{}{\mathsf{case}\ 0\ \mathsf{of}\ \{\mathsf{S} \to W \mid 0 \to Z\} \to_v \left\{(Z)^1\right\}}$$

A Probabilistic λ-calculus: Operational Semantics

$$\frac{}{M \oplus_p N \;\to_v\; \{\, M^p, N^{1-p} \,\}}$$

$$\frac{M \;\to_v\; \{\, L_i^{p_i} \mid i \in I \,\}}{\mathsf{let}\ x = M\ \mathsf{in}\ N \;\to_v\; \{\, (\mathsf{let}\ x = L_i\ \mathsf{in}\ N)^{p_i} \mid i \in I \,\}}$$

A Probabilistic λ-calculus: Operational Semantics

$$\frac{\mathscr{D} \overset{VD}{=} \left\{ M_j^{p_j} \;\middle|\; j \in J \right\} + \mathscr{D}_V \qquad \forall j \in J,\quad M_j \;\to_v\; \mathscr{E}_j}{\mathscr{D} \;\to_v\; \left(\sum_{j \in J} p_j \cdot \mathscr{E}_j\right) + \mathscr{D}_V}$$

For $\mathscr{D}$ a distribution of terms:

$$[\![\, \mathscr{D} \,]\!] \;=\; \sup_{n \in \mathbb{N}} \left(\left\{ \mathscr{E}_n \;\middle|\; \mathscr{D} \Rrightarrow_v^n \mathscr{E}_n \right\} \right)$$

where $\Rrightarrow_v^n$ is $\to_v^n$ followed by projection on values.

We let $[\![\, M \,]\!] \;=\; [\![\, \{ M^1 \} \,]\!]$.

M is AST iff $\sum [\![\, M \,]\!] \;=\; 1$.

Random Walks as Probabilistic Terms

- Biased random walk:

$$M_{bias} \;=\; \Big(\text{letrec } f \;=\; \lambda x.\text{case } x \text{ of } \Big\{ \mathsf{S} \to \lambda y.f(y) \oplus_{\frac{2}{3}} (f(\mathsf{S}\,\mathsf{S}\,y))) \;\mid\; 0 \to 0 \Big\}\Big)\ \underline{n}$$

- Unbiased random walk:

$$M_{unb} \;=\; \Big(\text{letrec } f \;=\; \lambda x.\text{case } x \text{ of } \Big\{ \mathsf{S} \to \lambda y.f(y) \oplus_{\frac{1}{2}} (f(\mathsf{S}\,\mathsf{S}\,y))) \;\mid\; 0 \to 0 \Big\}\Big)\ \underline{n}$$

$$\sum [\![\, M_{bias} \,]\!] \;=\; \sum [\![\, M_{unb} \,]\!] \;=\; 1$$

Capture this in a sized type system?

Another Term

We also want to capture terms as:

$$M_{nat} \;=\; \Big(\text{letrec } f \;=\; \lambda x.x \oplus_{\frac{1}{2}} \mathsf{S}\ (f\ x)\Big)\ 0$$

of semantics

$$[\![\, M_{nat} \,]\!] \;=\; \Big\{(0)^{\frac{1}{2}}, (\mathsf{S}\ 0)^{\frac{1}{4}}, (\mathsf{S}\ \mathsf{S}\ 0)^{\frac{1}{8}}, \ldots\Big\}$$

summing to 1.

(This is the geometric distribution.)

Distribution Types

A Probabilistic Counterpart to Sized Types

Beyond SN Terms, Towards Distribution Types

First idea: extend the sized type system with:

$$\text{Choice} \quad \frac{\Gamma \ \vdash \ M : \sigma \qquad \Gamma \ \vdash \ N : \sigma}{\Gamma \ \vdash \ M \oplus_p N \ : \ \sigma}$$

and "unify" types of M and N by subtyping.

Kind of product interpretation of $\oplus$: we can't capture more than SN...

Beyond SN Terms, Towards Distribution Types

First idea: extend the sized type system with:

$$\text{Choice} \quad \frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M \oplus_p N : \sigma}$$

and "unify" types of M and N by subtyping.

We get at best

$$f \ : \ \mathsf{Nat}^{\widehat{\widehat{\mathfrak{i}}}} \to \mathsf{Nat}^{\infty} \ \vdash \ \lambda y.f(y) \oplus_{\frac{1}{2}} (f(\mathsf{S}\,\mathsf{S}\,y))) \ : \ \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \mathsf{Nat}^{\infty}$$

and can't use a variation of the letrec rule on that.

Beyond SN Terms, Towards Distribution Types

We will use distribution types, built as follows:

$$\text{Choice} \quad \frac{\Gamma \mid \Theta \vdash M : \mu \qquad \Gamma \mid \Psi \vdash N : \nu \qquad \{\!|\, \mu \,|\!\} = \{\!|\, \nu \,|\!\}}{\Gamma \mid \Theta \oplus_p \Psi \vdash M \oplus_p N : \mu \oplus_p \nu}$$

Now

$$f \ : \ \left\{ \left(\mathsf{Nat}^{\mathfrak{i}} \to \mathsf{Nat}^{\infty}\right)^{\frac{1}{2}},\ \left(\mathsf{Nat}^{\widehat{\widehat{\mathfrak{i}}}} \to \mathsf{Nat}^{\infty}\right)^{\frac{1}{2}} \right\}$$

$$\vdash$$

$$\lambda y.f(y) \oplus_{\frac{1}{2}} (f(\mathsf{S}\,\mathsf{S}\,y))) \ : \ \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \mathsf{Nat}^{\infty}$$

Designing the Fixpoint Rule

$$f \; : \; \left\{ \left(\mathsf{Nat}^{\mathfrak{i}} \to \mathsf{Nat}^{\infty}\right)^{\frac{1}{2}}, \; \left(\mathsf{Nat}^{\widehat{\widehat{\mathfrak{i}}}} \to \mathsf{Nat}^{\infty}\right)^{\frac{1}{2}} \right\}$$

$$\vdash$$

$$\lambda y. f(y) \oplus_{\frac{1}{2}} (f(\mathsf{S}\,\mathsf{S}\,y))) \; : \; \color{red}{\mathsf{Nat}^{\widehat{\mathfrak{i}}}} \to \mathsf{Nat}^{\infty}$$

induces a random walk on $\mathbb{N}$:

- on $n+1$, move to n with probability $\frac{1}{2}$, on $n+2$ with probability $\frac{1}{2}$,
- on 0, loop.

The type system ensures that there is no recursive call from size 0.

Random walk AST ($=$ reaches 0 with proba 1) $\Rightarrow$ termination.

Designing the Fixpoint Rule

$$\text{LetRec}\quad \frac{\{\!|\,\Gamma\,|\!\} = \mathsf{Nat} \qquad \mathfrak{i} \notin \Gamma \text{ and } \mathfrak{i} \text{ positive in } \nu \qquad \left\{ (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{i}/\mathfrak{s}_j])^{p_j} \;\middle|\; j \in J \right\} \text{ induces an AST sized walk} \qquad \Gamma \,|\, f : \left\{ (\mathsf{Nat}^{\mathfrak{s}_j} \to \nu[\mathfrak{i}/\mathfrak{s}_j])^{p_j} \;\middle|\; j \in J \right\} \vdash V : \mathsf{Nat}^{\widehat{\mathfrak{i}}} \to \nu[\mathfrak{i}/\widehat{\mathfrak{i}}]}{\Gamma \,|\, \emptyset \vdash \mathsf{letrec}\ f = V : \mathsf{Nat}^{\mathfrak{r}} \to \nu[\mathfrak{i}/\mathfrak{r}]}$$

Sized walk: AST is checked by an external PTIME procedure.

Generalized Random Walks and the Necessity of Affinity

A crucial feature: our type system is affine.

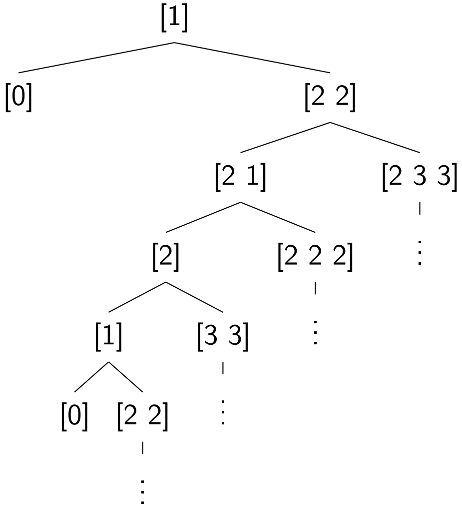
Higher-order symbols occur at most once. Consider:

$$M_{naff} \;=\; \text{letrec } f \;=\; \lambda x.\text{case } x \text{ of } \left\{ \mathsf{S} \to \lambda y.f(y) \oplus_{\frac{2}{3}} (f(\mathsf{S}\,\mathsf{S}\,y)\,;\, f(\mathsf{S}\,\mathsf{S}\,y)) \;\;\middle|\;\; 0 \to 0 \right\}$$

The induced sized walk is AST, but M_{naff} is not.

Generalized Random Walks and the Necessity of Affinity

Tree of recursive calls, starting from 1:



Leftmost edges have probability $\frac{2}{3}$; rightmost ones $\frac{1}{3}$.

This random process is not AST.

Problem: modelisation by sized walk only makes sense for affine programs.

Key Property I: Subject Reduction

Main idea: reduction of

$$\emptyset \mid \emptyset \vdash 0 \oplus 0 \;:\; \left\{ \left(\mathsf{Nat}^{\widehat{\mathfrak{s}}}\right)^{\frac{1}{2}}, \left(\mathsf{Nat}^{\widehat{\widehat{\mathfrak{r}}}}\right)^{\frac{1}{2}} \right\}$$

is to

$$\left\{ \left(0 \;:\; \mathsf{Nat}^{\widehat{\mathfrak{s}}}\right)^{\frac{1}{2}}, \left(0 \;:\; \mathsf{Nat}^{\widehat{\widehat{\mathfrak{r}}}}\right)^{\frac{1}{2}} \right\}$$

1. Same expectation type: $\frac{1}{2} \cdot \mathsf{Nat}^{\widehat{\mathfrak{s}}} + \frac{1}{2} \cdot \mathsf{Nat}^{\widehat{\widehat{\mathfrak{r}}}}$
2. Splitting of $[\![\, 0 \oplus 0 \,]\!]$ in a typed representation $\rightarrow$ notion of pseudo-representation

Key Property I: Subject Reduction

Theorem

Let $M \in \Lambda_{\oplus}$ be such that $\emptyset \,|\, \emptyset \vdash M \,:\, \mu$. Then there exists a closed typed distribution $\left\{ (W_j \,:\, \sigma_j)^{p'_j} \;\middle|\; j \in J \right\}$ such that

- $\mathbb{E}\left((W_j \,:\, \sigma_j)^{p'_j} \right) \preccurlyeq \mu$,
- *and that* $\left[(W_j)^{p'_j} \;\middle|\; j \in J \right]$ *is a pseudo-representation of* $[\![M]\!]$.

By the soundness theorem of next slide, this inequality is in fact an equality.

Key Property II: Typing Soundness

Theorem (Typing soundness)

If $\Gamma \mid \Theta \vdash M \, : \, \mu$, then M is AST.

Proof by reducibility, using set of candidates parametrized by probabilities.

Reducibility, the Probabilistic Case

Usual reducibility proof:

$$M \text{ closed of type } \sigma \quad \Rightarrow \quad M \in \mathit{Red}_\sigma \quad \Rightarrow \quad M \text{ is SN}$$

In our setting:

Reducibility, the Probabilistic Case

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In our setting:

$$M \in TRed_\sigma^p \quad \Rightarrow \qquad \sum [\![M]\!] \geq p$$

Reducibility, the Probabilistic Case

Usual reducibility proof:

$$M \text{ closed of type } \sigma \quad \Rightarrow \quad M \in Red_\sigma \quad \Rightarrow \quad M \text{ is SN}$$

In our setting:

$$M \text{ closed of type } \sigma \quad \Rightarrow \quad \forall p < 1,\ M \in TRed_\sigma^p \quad \Rightarrow \quad \forall p < 1,\ \sum [\![\, M \,]\!] \geq p$$

p increases with the number of fixpoint unfoldings we do, and we prove that M is in $TRed_\sigma^p$ iff its n-unfolding is.

Reducibility, the Probabilistic Case

Usual reducibility proof:

M closed of type $\sigma \quad \Rightarrow \quad M \in Red_\sigma \quad \Rightarrow \quad M$ is SN

In our setting:

M closed of type $\sigma \quad \Rightarrow \quad M \in TRed_\sigma^1 \quad \Rightarrow \quad \sum [\![M]\!] = 1$ i.e. M AST

by a continuity lemma.

Reducibility, the Probabilistic Case – Open Terms

Usual case: $\quad \overrightarrow{x} : \overrightarrow{\sigma} \vdash M : \tau \quad \Rightarrow \quad \forall \overrightarrow{V} \in \overrightarrow{VRed_{\sigma}}, \;\; M[\overrightarrow{x} / \overrightarrow{V}] \in Red_{\tau}$

Reducibility, the Probabilistic Case – Open Terms

Usual case: $\overrightarrow{x} : \overrightarrow{\sigma} \vdash M : \tau \quad \Rightarrow \quad \forall \overrightarrow{V} \in \overrightarrow{VRed_{\sigma}}, \;\; M[\overrightarrow{x}/\overrightarrow{V}] \in Red_{\tau}$

In our setting: if $\Gamma \mid y : \{\tau_j^{p_j}\}_{j \in J} \vdash M : \mu$ then

- $\forall (q_i)_i \in [0,1]^n, \;\; \forall \overrightarrow{V} \in \prod_{i=1}^n \; \mathsf{VRed}_{\sigma_i}^{q_i},$
- $\forall \left(q'_j\right)_j \in [0,1]^J, \;\; \forall W \in \bigcap_{j \in J} \; \mathsf{VRed}_{\tau_j}^{q'_j},$
- we have $M[\overrightarrow{x}, y/\overrightarrow{V}, W] \in \mathsf{TRed}_{\mu}^{\alpha}$

where $\alpha \;\; = \;\; (\prod_{i=1}^n q_i)\left(\left(\sum_{j \in J} p_j q'_j\right) + 1 - \left(\sum_{j \in J} p_j\right)\right).$

Another Approach Using Dependent Types

Another Approach Using Dependent Types

Alternative approach to sized types: dependent types.

See Xi (2002), *Dependent Types for Program Termination Verification*.

Examples of dependent types à la Xi:

- $\varphi \mid \Gamma \vdash \underline{2} : \mathit{int}\,(2)$
- $\varphi \mid \Gamma \vdash \left\langle 2 \mid \underline{2} \right\rangle : \Sigma a : \mathit{int}.\, \mathit{int}\,(a)$

Terms of base type: annotated with size information which can be packed in the term (annotation by a size expression). Produces a sum type (existential).

φ: context of constraints on free size variables, like $a \in \{a \in \mathit{int} \mid a > 2\}$.

Another Approach Using Dependent Types

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See Xi (2002), *Dependent Types for Program Termination Verification*.

Examples of dependent types à la Xi:

- $\varphi \mid \Gamma \vdash + \;:\; \Pi\{a : int,\ b : int\}.\, int\,(a) \times int\,(b) \to int\,(a+b)$
- $\varphi \mid \Gamma \vdash \times \;:\; \Pi\{a : int,\ b : int\}.\, int\,(a) \times int\,(b) \to int\,(a \times b)$

Functions typically have universally quantified arguments (product type). Note that we could derive terms from $+$ and $\times$ which use sum types for return types.

Another Approach Using Dependent Types

Sum types allow to get a uniform Choice rule:

$$\frac{\varphi \,|\, \Gamma \,|\, \Theta \vdash M \,:\, \sigma \qquad \varphi \,|\, \Gamma \,|\, \Theta \vdash N \,:\, \sigma}{\varphi \,|\, \Gamma \,|\, \Theta \vdash M \oplus_p N \,:\, \sigma}$$

No longer need for distribution types!

Various sizes are annotated in the term.

Another Approach Using Dependent Types

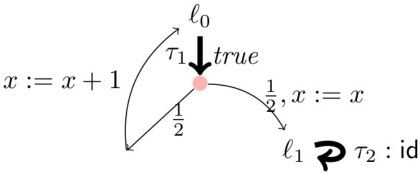
$$\frac{\begin{array}{c}\{\!|\, \Gamma \,|\!\} \subseteq \{\textbf{bool}, \textbf{int}\} \\ \varphi, \overrightarrow{a} : \overrightarrow{\gamma} \mid \Gamma \mid f : \prod \overrightarrow{a} : \overrightarrow{\gamma}.\sigma \Vdash_{\mathcal{P}} V : \sigma \\ \mathsf{letrec}\,(\mathcal{P}, \rho) \text{ is AST for every } \rho \models \varphi\end{array}}{\varphi \mid \Gamma \mid \Theta \vdash \mathsf{letrec}\ f[\overrightarrow{a} : \overrightarrow{\gamma}] : \sigma \ = \ V : \prod \overrightarrow{a} : \overrightarrow{\gamma}.\sigma}$$

- Rely on PTS analysis (more general than random walks)
- letrec enters a new mode: typing relation $\Vdash_{\mathcal{P}}$ indexed by a PTS

PTS = probabilistic transition system

Examples of PTS

```
int x := 0;
while (flip (0.5))
    x ++;
// end
```



Examples from Chakarov and Sankaranarayanan (2013), *Probabilistic Program Analysis with Martingales*

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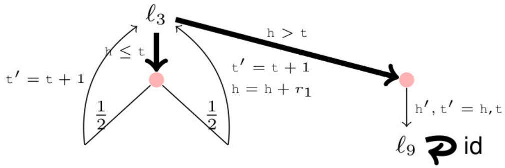

Examples from Chakarov and Sankaranarayanan (2013), *Probabilistic Program Analysis with Martingales*

Our point: replaced sized walks by these processes modeling the flow of recursive calls. The process is built on-the-fly by the type system.

Building the PTS

$$\frac{\varphi \vdash \overrightarrow{I} : \overrightarrow{\gamma} \qquad \mathcal{P} = \textbf{leaf}\left(f\left[\overrightarrow{a} \mapsto \overrightarrow{I}\right]\right)}{\varphi \,|\, \Gamma \,|\, f : \prod \overrightarrow{a} : \overrightarrow{\gamma}.\sigma \Vdash_{\mathcal{P}} f\left[\overrightarrow{I}\right] : \sigma[\overrightarrow{a} / \overrightarrow{I}]}$$

$\textbf{leaf}\left(f\left[\overrightarrow{a} \mapsto \overrightarrow{I}\right]\right)$ is a PTS with just one node, looping on itself and updating $\overrightarrow{a}$ with $\overrightarrow{[\![I]\!]_\rho}$.

Building the PTS

$$\frac{\varphi \mid \Gamma \mid \emptyset \vdash M : \mathbf{bool}(I) \qquad \varphi,\, I = 1 \mid \Delta \mid \Theta \Vdash_{\mathcal{P}} N : \sigma \qquad \varphi,\, I = 0 \mid \Delta \mid \Theta \Vdash_{\mathcal{Q}} L : \sigma}{\varphi \mid \Gamma,\, \Delta \mid \Theta \Vdash_{\mathsf{if}(I,\mathcal{P},\mathcal{Q})} \mathsf{if}\ M\ \mathsf{then}\ N\ \mathsf{else}\ L : \sigma}$$

if $(I, \mathcal{P}, \mathcal{Q})$ is a PTS containing $\mathcal{P}$ and $\mathcal{Q}$ and with one new node branching to the root of $\mathcal{P}$ or of $\mathcal{Q}$ depending on $[\![\, I \,]\!]_\rho$.

Building the PTS

$$\frac{\varphi \,|\, \Gamma \,|\, \Theta \ \Vdash_{\mathcal{P}} \ M : \sigma \qquad \varphi \,|\, \Gamma \,|\, \Theta \ \Vdash_{\mathcal{Q}} \ N : \sigma}{\varphi \,|\, \Gamma \,|\, \Theta \ \Vdash_{\mathcal{P} \oplus_p \mathcal{Q}} \ M \oplus_q N : \sigma}$$

$\mathcal{P} \oplus_p \mathcal{Q}$ is a PTS containing $\mathcal{P}$ and $\mathcal{Q}$ and with one new node branching to the root of $\mathcal{P}$ or of $\mathcal{Q}$ depending on a biased coin flip of probability q.

Building the PTS

$$\frac{\{\!|\, \Gamma \,|\!\} \subseteq \{\mathbf{bool}, \mathbf{int}\} \qquad \varphi, \overrightarrow{a} : \overrightarrow{\gamma} \mid \Gamma \mid f : \prod \overrightarrow{a} : \overrightarrow{\gamma}.\sigma \Vdash_{\mathcal{P}} V : \sigma \qquad \mathsf{letrec}\,(\mathcal{P}, \rho) \text{ is AST for every } \rho \models \varphi}{\varphi \mid \Gamma \mid \Theta \vdash \mathsf{letrec}\ f[\overrightarrow{a} : \overrightarrow{\gamma}] : \sigma \ = \ V : \prod \overrightarrow{a} : \overrightarrow{\gamma}.\sigma}$$

letrec $(\mathcal{P}, \rho)$ is a PTS obtained from $\mathcal{P}$ by making the loops on the leaves pointing to the root of $\mathcal{P}$.

Conjecture

We have strong hints that:

$$M \text{ has type } \sigma \quad \Rightarrow \quad M \text{ is AST.}$$

(we have a proof sketch based on the previous realizability argument).

Note that the system is again affine.

Conclusion

First type system:

- Affine type system with distributions of types
- Sized walks induced by the letrec rule and solved by an external PTIME procedure
- Subject reduction + soundness for AST

Second type system:

- Finer analysis: more expressive sizes, modelization by PTS
- No need for distribution types thanks to sum types
- Still affine
- Soundness is work in progress

Thank you for your attention!

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