Semantics of linear logic and higher-order model-checking

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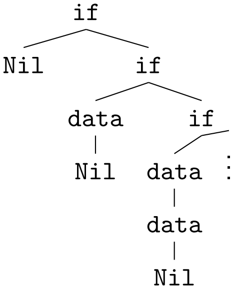
Model-checking higher-order programs

For higher-order programs with recursion, the model $\mathcal{M}$ of interest is a higher-order regular tree.

Example:

$$
\begin{array}{rcl}
\texttt{Main} & = & \texttt{Listen Nil} \\
\texttt{Listen } x & = & \texttt{if } \mathit{end} \texttt{ then } x \texttt{ else Listen (data } x\texttt{)}
\end{array}
$$

modelled as



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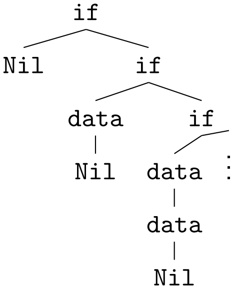


Finite representation: HORS

Model-checking higher-order programs

For higher-order programs with recursion, the model $\mathcal{M}$ of interest is a higher-order regular tree

over which we run

an alternating parity tree automaton (APT) $\mathcal{A}_\varphi$

corresponding to a

monadic second-order logic (MSO) formula φ.

(safety, liveness properties, etc)

Can we decide whether a higher-order regular tree satisfies a MSO formula?

Higher-order recursion schemes

Some regularity for infinite trees

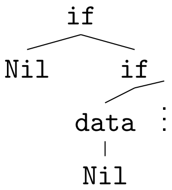
Higher-order recursion schemes

$$\begin{array}{rcl} \texttt{Main} & = & \texttt{Listen Nil} \\ \texttt{Listen}\ x & = & \texttt{if}\ \textit{end}\ \texttt{then}\ x\ \texttt{else Listen (data}\ x\texttt{)} \end{array}$$

is abstracted as

$$\mathcal{G} = \begin{cases} \texttt{S} & = \quad \texttt{L Nil} \\ \texttt{L}\ x & = \quad \texttt{if}\ x\ (\texttt{L}\ (\texttt{data}\ x\ )\ ) \end{cases}$$

which produces the higher-order tree of actions



Higher-order recursion schemes

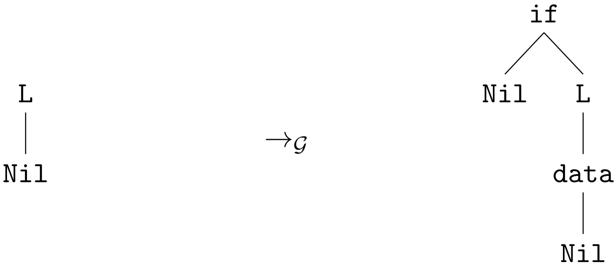
$$\mathcal{G} \quad = \quad \left\{ \begin{array}{lcl} \mathtt{S} & = & \mathtt{L}\ \mathtt{Nil} \\ \mathtt{L}\ x & = & \mathtt{if}\ x\ (\mathtt{L}\ (\mathtt{data}\ x\ )\ ) \end{array} \right.$$

Rewriting starts from the start symbol S:

$$\mathtt{S} \quad \rightarrow_{\mathcal{G}} \quad \begin{array}{c} \mathtt{L} \\ | \\ \mathtt{Nil} \end{array}$$

Higher-order recursion schemes

$$\mathcal{G} \;=\; \begin{cases} \mathtt{S} & = \;\; \mathtt{L}\ \mathtt{Nil} \\ \mathtt{L}\ x & = \;\; \mathtt{if}\ x\ (\mathtt{L}\ (\mathtt{data}\ x\ )\ ) \end{cases}$$



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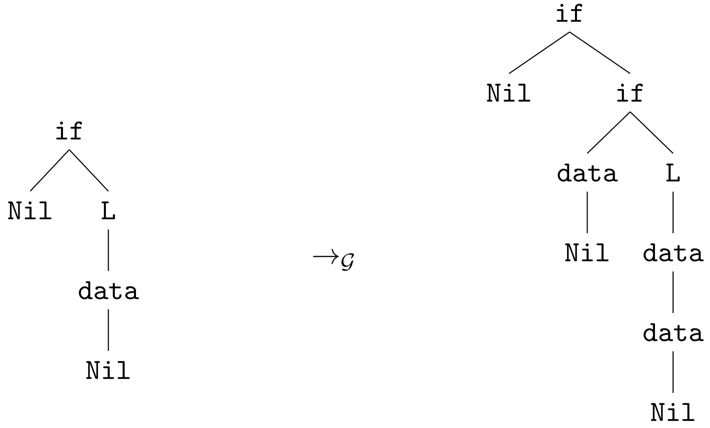
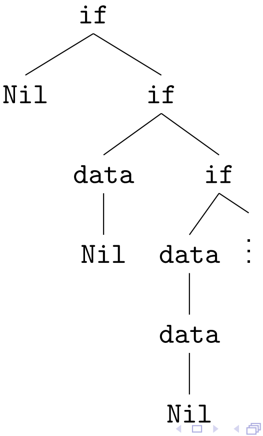


Higher-order recursion schemes

$$\mathcal{G} = \begin{cases} \texttt{S} & = \quad \texttt{L Nil} \\ \texttt{L}\ x & = \quad \texttt{if}\ x\ (\texttt{L}\ (\texttt{data}\ x\ )\ ) \end{cases}$$

$\langle \mathcal{G} \rangle \quad =$



Higher-order recursion schemes

$$\mathcal{G} \;=\; \begin{cases} \mathtt{S} & = \quad \mathtt{L}\ \mathtt{Nil} \\ \mathtt{L}\ x & = \quad \mathtt{if}\ x\ (\mathtt{L}\ (\mathtt{data}\ x\ )\ ) \end{cases}$$

"Everything" is simply-typed, and

well-typed programs can't go too wrong:

we can detect productivity, and enforce it (replace divergence by outputing a distinguished symbol Ω in one step).

HORS can alternatively be seen as simply-typed λ-terms with

simply-typed recursion operators $Y_\sigma \;:\; (\sigma \rightarrow \sigma) \rightarrow \sigma$.

Alternating parity tree automata

Alternating parity tree automata

For a MSO formula φ,

$$\langle \mathcal{G} \rangle \quad \vDash \quad \varphi$$

iff an equivalent APT $\mathcal{A}_\varphi$ has a run over $\langle \mathcal{G} \rangle$.

APT $=$ alternating tree automata (ATA) $+$ parity condition.

Alternating tree automata

ATA: non-deterministic tree automata whose transitions may duplicate or drop a subtree.

Typically: $\delta(q_0, \texttt{if}) \ = \ (2, q_0) \wedge (2, q_1)$.

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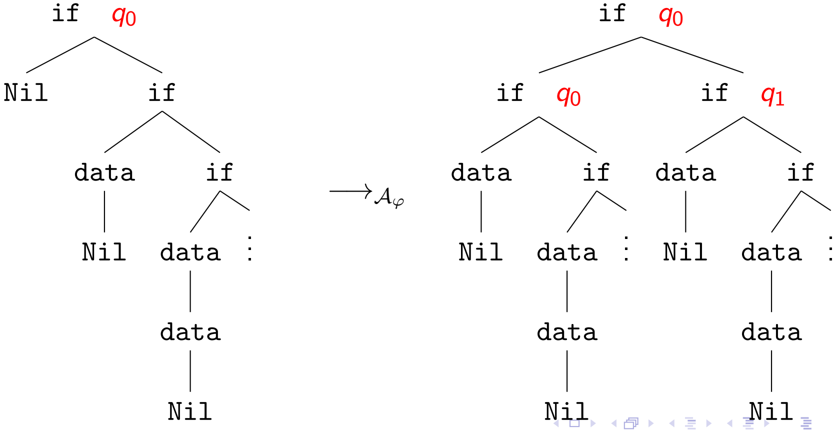
Typically: $\delta(q_0, \texttt{if}) \;=\; (2, q_0) \wedge (2, q_1)$.



Alternating parity tree automata

MSO discriminates inductive from coinductive behaviour.

This allows to express properties as

"a given operation is executed infinitely often in some execution"

or

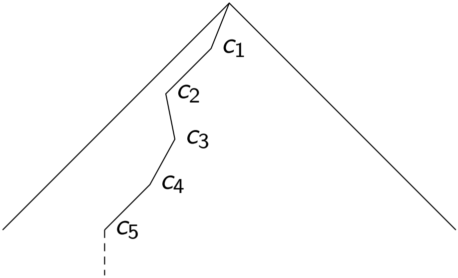
"after a `read` operation, a `write` eventually occurs".

Alternating parity tree automata

Each state of an APT is attributed a color

$$\Omega(q) \in Col \subseteq \mathbb{N}$$

An infinite branch of a run-tree is winning iff the maximal color among the ones occuring infinitely often along it is even.



Alternating parity tree automata

Each state of an APT is attributed a color

$$\Omega(q) \in Col \subseteq \mathbb{N}$$

An infinite branch of a run-tree is winning iff the maximal color among the ones occuring infinitely often along it is even.

A run-tree is winning iff all its infinite branches are.

For a MSO formula φ:

$$\mathcal{A}_\varphi \text{ has a winning run-tree over } \langle \mathcal{G} \rangle \text{ iff } \langle \mathcal{G} \rangle \vDash \phi.$$

Intersection types and alternation

Alternating tree automata and intersection types

A key remark (Kobayashi 2009):

$$\delta(q_0, \texttt{if}) \;=\; (2, q_0) \wedge (2, q_1)$$

can be seen as the intersection typing

$$\texttt{if} \;:\; \emptyset \rightarrow (q_0 \wedge q_1) \rightarrow q_0$$

refining the simple typing

$$\texttt{if} \;:\; o \rightarrow o \rightarrow o$$

(this talk is NOT about filter models!)

Alternating tree automata and intersection types

In a derivation typing $\texttt{if} \;\; T_1 \;\; T_2$:

$$\text{App}\ \dfrac{\text{App}\ \dfrac{\delta\ \dfrac{}{\emptyset \vdash \texttt{if} : \emptyset \to (q_0 \wedge q_1) \to q_0} \qquad \color{red}{\emptyset}}{\emptyset \vdash \texttt{if}\ T_1 : (q_0 \wedge q_1) \to q_0} \qquad \color{red}{\dfrac{\vdots}{\Gamma_{21} \vdash T_2 : q_0}} \qquad \color{red}{\dfrac{\vdots}{\Gamma_{22} \vdash T_2 : q_1}}}{\Gamma_{21}, \Gamma_{22} \vdash \texttt{if}\ T_1\ T_2 : q_0}$$

Intersection types naturally lift to higher-order – and thus to $\mathcal{G}$, which finitely represents $\langle \mathcal{G} \rangle$.

Theorem (Kobayashi)

$S : q_0 \vdash S : q_0$ *iff* *the ATA $\mathcal{A}_\varphi$ has a run-tree over $\langle \mathcal{G} \rangle$.*

A type-system for verification: without parity conditions

$$\text{Axiom} \qquad \frac{}{x : \bigwedge_{\{i\}} \theta_i :: \kappa \ \vdash \ x : \theta_i :: \kappa}$$

$$\delta \qquad \frac{\{\, (i, q_{ij}) \mid 1 \leq i \leq n, 1 \leq j \leq k_i \} \quad \text{satisfies} \quad \delta_A(q, a)}{\emptyset \vdash a : \bigwedge_{j=1}^{k_1} q_{1j} \rightarrow \ \dots \ \rightarrow \bigwedge_{j=1}^{k_n} q_{nj} \rightarrow q :: o \rightarrow \cdots \rightarrow o}$$

$$\text{App} \qquad \frac{\Delta \vdash t : (\ \theta_1 \ \wedge \cdots \wedge \ \theta_k) \rightarrow \theta :: \kappa \rightarrow \kappa' \qquad \Delta_i \vdash u : \theta_i :: \kappa}{\Delta, \Delta_1, \dots, \Delta_k \ \vdash \ t\, u : \theta :: \kappa'}$$

$$\lambda \qquad \frac{\Delta, x : \bigwedge_{i \in I} \theta_i :: \kappa \ \vdash \ t : \theta :: \kappa'}{\Delta \ \vdash \ \lambda x . t : \left(\bigwedge_{i \in I} \theta_i\right) \rightarrow \theta :: \kappa \rightarrow \kappa'}$$

$$\mathit{fix} \qquad \frac{\Gamma \vdash \mathcal{R}(F) : \theta :: \kappa}{\overline{F : \theta :: \kappa \ \vdash \ F : \theta :: \kappa}}$$

A closer look at the Application rule

$$\text{App} \qquad \frac{\Delta \vdash t : (\ \theta_1\ \wedge \cdots \wedge\ \theta_k) \rightarrow \theta\ ::\ \kappa \rightarrow \kappa' \qquad \Delta_i \vdash u\ :\ \theta_i\ ::\ \kappa}{\Delta\,,\,\Delta_1\,,\,\ldots\,,\,\Delta_k\ \vdash\ t\,u\ :\ \theta :: \kappa'}$$

can be decomposed as:

$$\frac{\Delta\ \vdash\ t\ :\ (\ \bigwedge_{i=1}^{n}\ \theta_i\ ) \rightarrow \theta' \qquad \dfrac{\Delta_i\ \vdash\ u\ :\ \theta_i \qquad \forall i \in \{1, \ldots\ n\}}{\Delta_1, \ldots, \Delta_n\ \vdash\ u\ :\ \bigwedge_{i=1}^{n}\ \theta_i}\ \text{Right}\ \bigwedge}{\Delta,\ \Delta_1, \ldots, \Delta_n\ \vdash\ t\ u\ :\ \theta'}$$

A closer look at the Application rule

$$\dfrac{\Delta \ \vdash \ t : (\bigwedge_{i=1}^{n} \ \theta_i) \rightarrow \theta' \qquad \dfrac{\Delta_i \ \vdash \ u : \theta_i \qquad \forall i \in \{1, \dots n\}}{\Delta_1, \dots, \Delta_n \ \vdash \ u : \bigwedge_{i=1}^{n} \ \theta_i} \ \text{Right} \ \bigwedge}{\Delta, \Delta_1, \dots, \Delta_n \ \vdash \ t \ u : \theta'}$$

Linear decomposition of the intuitionnistic arrow:

$$A \Rightarrow B \ = \ !A \multimap B$$

Two steps: duplication / erasure, then linear use.

Right $\bigwedge$ corresponds to the Promotion rule of indexed linear logic.

Intersection types and semantics of linear logic

$$A \Rightarrow B \quad = \quad !\, A \multimap B$$

Two interpretations of the exponential modality:

Qualitative models
(Scott semantics)

$$!\, A \quad = \quad \mathcal{P}_{fin}(A)$$

$$[\![o \Rightarrow o]\!] \; = \; \mathcal{P}_{fin}(Q) \times Q$$

$$\{q_0,\, q_0,\, q_1\} \quad = \quad \{q_0,\, q_1\}$$

Order closure

Quantitative models
(Relational semantics)

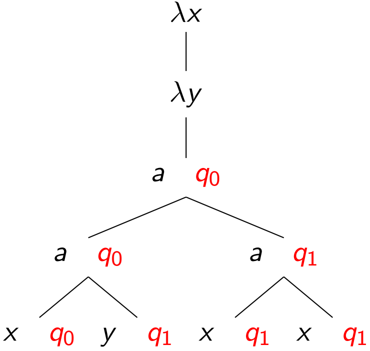
$$!\, A \quad = \quad \mathcal{M}_{fin}(A)$$

$$[\![o \Rightarrow o]\!] \; = \; \mathcal{M}_{fin}(Q) \times Q$$

$$[q_0,\, q_0,\, q_1] \quad \neq \quad [q_0,\, q_1]$$

Unbounded multiplicities

An example of interpretation



In *Rel*, one denotation:

$$([q_0, q_1, q_1], [q_1], q_0)$$

In *ScottL*, a set containing the principal type

$$(\{q_0, q_1\}, \{q_1\}, q_0)$$

but also

$$(\{q_0, q_1, q_2\}, \{q_1\}, q_0)$$

and

$$(\{q_0, q_1\}, \{q_0, q_1\}, q_0)$$

and . . .

Intersection types and semantics of linear logic

```tikzcd
Rel_! \arrow[r, leftrightarrow, "Bucciareli-Ehrhard", "de\ Carvalho"'] \arrow[d, "Ehrhard"'] & \text{Non-idempotent types} \arrow[d, "Ehrhard"] \\
ScottL_! \arrow[r, leftrightarrow, "Ehrhard", "Terui"'] & \text{Idempotent types}
```

Fundamental idea:

$$[\![t]\!] \quad \cong \quad \{\, \theta \mid \emptyset \vdash t : \theta \,\}$$

for a closed term.

Intersection types and semantics of linear logic

$$
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\end{tikzcd}
$$

Let t be a term normalizing to a tree $\langle t \rangle$ and $\mathcal{A}$ be an alternating automaton.

$$\mathcal{A} \text{ accepts } \langle t \rangle \text{ from } q \quad \Leftrightarrow \quad q \in [\![t]\!] \quad \Leftrightarrow \quad \emptyset \vdash t : q :: o$$

Extension with recursion and parity condition?

Adding parity conditions to the type system

Alternating parity tree automata

We add coloring annotations to intersection types:

$$\delta(q_0, \mathtt{if}) \;=\; (2, q_0) \wedge (2, q_1)$$

now corresponds to

$$\mathtt{if} \;:\; \emptyset \rightarrow \left(\Box_{\Omega(q_0)}\, q_0 \wedge \Box_{\Omega(q_1)}\, q_1\right) \rightarrow q_0$$

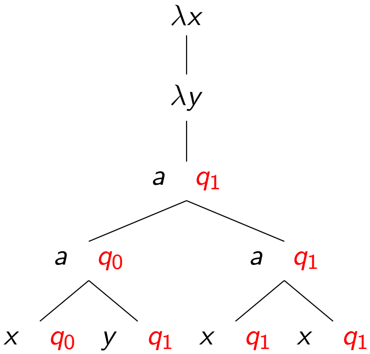
Idea: `if` is a run-tree with two holes:

$$\begin{array}{c} \mathtt{if} \\ \diagup \quad \diagdown \\ [\,]_{q_0} \quad [\,]_{q_1} \end{array}$$

A new neutral color: ϵ for an empty run-tree context $[\,]_q$.

An example of colored intersection type

Set $\Omega(q_i) \; = \; i$.



has type

$$\Box_0 \, q_0 \wedge \Box_1 \, q_1 \rightarrow \Box_1 \, q_1 \rightarrow q_1$$

Note the color 0 on q_0...

A type-system for verification (Grellois-Melliès 2014)

Axiom $$\frac{}{x \,:\, \bigwedge_{\{i\}} \square_{\epsilon}\ \theta_i\ ::\ \kappa\ \vdash\ x\,:\,\theta_i\ ::\ \kappa}$$

δ $$\frac{\{\,(i, q_{ij}) \mid 1 \leq i \leq n, 1 \leq j \leq k_i\,\}\quad \text{satisfies}\quad \delta_A(q, a)}{\emptyset \vdash a\,:\, \bigwedge_{j=1}^{k_1} \square_{\Omega(q_{1j})}\ q_{1j}\ \rightarrow\ \ldots\ \rightarrow\ \bigwedge_{j=1}^{k_n} \square_{\Omega(q_{nj})}\ q_{nj} \rightarrow q\ ::\ o \rightarrow \cdots \rightarrow o \rightarrow o}$$

App $$\frac{\Delta \vdash t : (\square_{m_1}\ \theta_1\ \wedge \cdots \wedge \square_{m_k}\ \theta_k) \rightarrow \theta\ ::\ \kappa \rightarrow \kappa' \qquad \Delta_i \vdash u\ :\ \theta_i\ ::\ \kappa}{\Delta\ +\ \square_{m_1}\Delta_1\ +\ \ldots\ +\ \square_{m_k}\Delta_k\ \vdash\ t\,u\ :\ \theta ::\ \kappa'}$$

fix $$\frac{\Gamma \vdash \mathcal{R}(F)\ :\ \theta\ ::\ \kappa}{F\ :\ \square_{\epsilon}\ \theta\ ::\ \kappa\ \vdash\ F\ :\ \theta\ ::\ \kappa}$$

λ $$\frac{\Delta\,,\, x\ :\ \bigwedge_{i\in I} \square_{m_i}\ \theta_i\ ::\ \kappa\ \vdash\ t\ :\ \theta\ ::\ \kappa'}{\Delta\ \vdash\ \lambda\, x\,.\,t\ :\ \left(\bigwedge_{i\in I} \square_{m_i}\ \theta_i\right) \rightarrow \theta ::\ \kappa \rightarrow \kappa'}$$

A type-system for verification

A colored Application rule:

$$\text{App} \quad \frac{\Delta \vdash t : (\Box_{c_1}\ \theta_1\ \wedge \cdots \wedge \Box_{c_k}\ \theta_k) \rightarrow \theta\ ::\ \kappa \rightarrow \kappa' \qquad \Delta_i \vdash u\ :\ \theta_i\ ::\ \kappa}{\Delta\ +\ \Box_{c_1}\Delta_1\ +\ \ldots\ +\ \Box_{c_k}\Delta_k\ \ \vdash\ \ t\,u\ :\ \theta :: \ \kappa'}$$

inducing a winning condition on infinite proofs: the node

$$\Delta_i \vdash u\ :\ \theta_i\ ::\ \kappa$$

has color c_i, others have color ϵ, and we use the parity condition.

A type-system for verification

We now capture all MSO:

Theorem (G.-Melliès 2014, from Kobayashi-Ong 2009)

$S : q_0 \vdash S : q_0$ admits a winning typing derivation iff the alternating parity automaton $\mathcal{A}$ has a winning run-tree over $\langle \mathcal{G} \rangle$.

We obtain decidability by considering idempotent types.

Non-idempotency is very helpful for proofs, but leads to infinitary constructions.

Colored models of linear logic

A closer look at the Application rule

$$\frac{\Delta \vdash t : (\Box_{m_1}\ \theta_1\ \wedge \cdots \wedge \Box_{m_k}\ \theta_k) \to \theta\ ::\ \kappa \to \kappa' \qquad \Delta_i \vdash u\ :\ \theta_i\ ::\ \kappa}{\Delta\ +\ \Box_{m_1}\Delta_1\ +\ \ldots\ +\ \Box_{m_k}\Delta_k\ \vdash\ t\,u\ :\ \theta :: \kappa'}$$

can be decomposed as:

$$\frac{\Delta \vdash t : (\bigwedge_{i=1}^{n}\ \Box_{m_i}\ \theta_i) \to \theta \qquad \dfrac{\dfrac{\Delta_1 \vdash u\ :\ \theta_1}{\Box_{m_1}\ \Delta_1\ \vdash\ u\ :\ \Box_{m_1}\ \theta_1} \quad \ldots \quad \dfrac{\Delta_n \vdash u\ :\ \theta_n}{\Box_{m_n}\ \Delta_n\ \vdash\ u\ :\ \Box_{m_n}\ \theta_1}}{\Box_{m_1}\Delta_1,\ \ldots,\ \Box_{m_n}\Delta_n\ \vdash\ u\ :\ \bigwedge_{i=1}^{n}\ \Box_{m_i}\ \theta_i}}{\Delta,\ \Box_{m_1}\Delta_1,\ \ldots,\ \Box_{m_n}\Delta_n\ \vdash\ t\,u\ :\ \theta}$$

Right $\Box$
Right $\bigwedge$

Right $\Box$ looks like a promotion. In linear logic:

$$A \Rightarrow B\ =\ !\,\Box\, A \multimap B$$

Our reformulation of the Kobayashi-Ong type system shows that $\Box$ is a modality (in the sense of S4) which distributes with the exponential in the semantics.

Colored semantics

We extend:

- *Rel* with countable multiplicites, coloring and an inductive-coinductive fixpoint
- *ScottL* with coloring and an inductive-coinductive fixpoint.

Methodology: think in the relational semantics, and adapt to the Scott semantics using Ehrhard's 2012 result:

the finitary model *ScottL* is the extensional collapse of *Rel*.

Infinitary relational semantics

Extension of *Rel* with infinite multiplicities:

$$\lightning\, A \quad = \quad \mathcal{M}_{count}(A)$$

and coloring modality

$$\Box\, A \quad = \quad Col \times A$$

Distributive law:

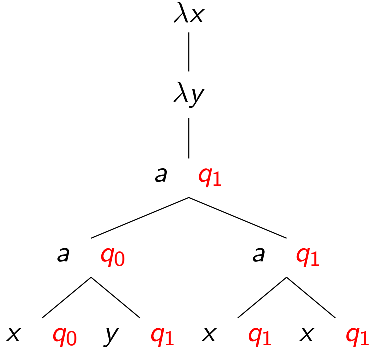
$$\begin{array}{rcl} \lambda_A & = & :\ \lightning\, \Box\, A \rightarrow \Box\, \lightning\, A \\ & & \{([(c, a_1),\ (c, a_2), \ldots], (c,\ [a_1,\ a_2, \ldots]))\ \mid a_i \in A,\ c \in Col\} \end{array}$$

Allows to compose comonads: $\lightning = \lightning\, \Box$ is an exponential in the infinitary relational semantics.

This induces a colored CCC $Rel_{\lightning}$ ($\rightarrow$ model of the λ-calculus).

An example of interpretation

Set $\Omega(q_i) = i$.



has denotation

$$([(0, q_0), (1, q_1), (1, q_1)], [(1, q_1)], q_1)$$

(corresponding to the type $\Box_0\, q_0 \wedge \Box_1\, q_1 \rightarrow \Box_1\, q_1 \rightarrow q_1$)

An inductive-coinductive fixpoint operator

$\mathbf{Y}$ transports

$$f \quad : \quad ⺓ X \otimes ⺓ A \multimap A$$

into

$$\mathbf{Y}_{X,A}(f) \quad : \quad ⺓ X \multimap A.$$

by composing together denotations of f in a way which satisfies the parity condition.

$\mathbf{Y}$ is a Conway operator, and $Rel_{⺓}$ is a model of the λY-calculus.

Model-checking and infinitary semantics

Conjecture

An APT $\mathcal{A}$ has a winning run from q_0 over $\langle \mathcal{G} \rangle$ if and only if

$$q_0 \in [\![\lambda(\mathcal{G})]\!]$$

where $\lambda(\mathcal{G})$ is a λY-term corresponding to $\mathcal{G}$.

Using Church encoding, we can also design an interpretation independent of the automaton of interest.

Finitary semantics

In ScottL, we define $\Box$, λ and $\mathbf{Y}$ similarly (using downward-closures).

$ScottL_{\lightning}$ is a model of the λY-calculus.

Theorem

An APT $\mathcal{A}$ has a winning run from q_0 over $\langle \mathcal{G} \rangle$ if and only if

$$q_0 \in [\![\lambda(\mathcal{G})]\!]$$

Corollary

The higher-order model-checking problem is decidable.

Conclusion

- Connections between intersection types and linear logic
- Refinement of the Kobayashi-Ong type system: coloring is a modality
- Colored models of the λY-calculus coming from linear logic
- Decidability using the finitary Scott semantics
- Raises interesting questions in semantics: infinitary models, coeffects...
- Ongoing work: a probabilistic extension

Thank you for your attention!

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