Verification by typing

Charles Grellois
joint work with Paul-André Melliès

PPS & LIAFA

25 juin 2014

Model-checking

Usual approach in verification: model-checking (Clarke, Emerson).
Interaction of a program and a property.

How do we model them ?

Many possible answers depending on the kind of program and property. A general approach would be undecidable. . .

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In this work we are concerned with higher-order programs: a function may take a function as input.

Example: `compose` $\phi\ x \;=\; \phi(\phi(x))$

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Higher-order recursion schemes

Idea:

- LIAFA-style: it is a kind of grammar whose parameters may be functions and which generates trees.
- PPS-style: it is a formalism equivalent to λY calculus with uninterpreted constants from a ranked alphabet.

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A silly functional program

$$\begin{array}{rcl} \texttt{Main} & = & \texttt{Listen Nil} \\ \texttt{Listen } x & = & \texttt{if } \mathit{end} \texttt{ then } x \texttt{ else Listen (data } x\texttt{)} \end{array}$$

With a recursion scheme we can model this program and produce its tree of behaviours.

Note that constants are not interpreted: in particular, a recursion scheme does not evaluate a `if`. We shall see that the problem is already pretty complex without this kind of additional reduction rules. . .

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$$\begin{array}{rcl} \mathtt{S} & = & \mathtt{L\ Nil} \\ \mathtt{L}\ x & = & \mathtt{if}\ x\ (\mathtt{L}\ (\mathtt{data}\ x\ )) \end{array}$$

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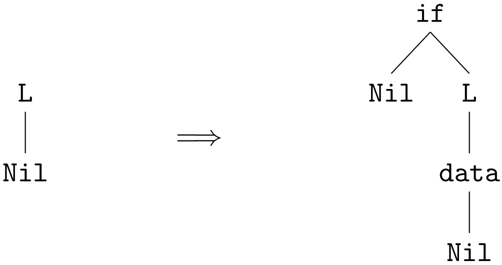

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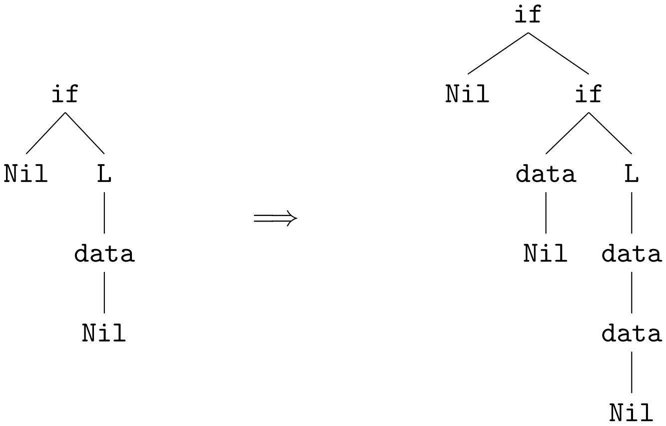

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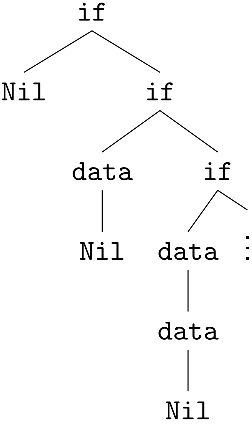
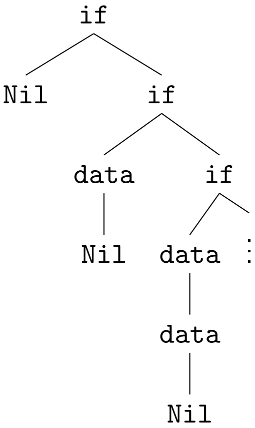


Silly program, but not regular tree…

Value tree of a recursion scheme



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Another recursion scheme

The previous recursion scheme was of order 1.

Indeed, the non-terminals are typed according to the ranked alphabet of constants.

We had $S : o$ and $L : o \to o$, of order 0 and 1 respectively. Their maximum is the order of the scheme.

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Another recursion scheme

An order-2 example (from Serre et al.):

$$\begin{array}{rcl} S & = & M\ \texttt{Nil} \\ M\ x & = & \texttt{if}\ (\ \mathit{commit}\ x\ )\ (\ A\ x\ M\ ) \\ A\ y\ \phi & = & \texttt{if}\ (\ \phi\ (\ \mathit{error}\ \mathit{end}\ )\ )\ (\ \phi\ (\ \mathit{cons}\ y\ )\ ) \end{array}$$

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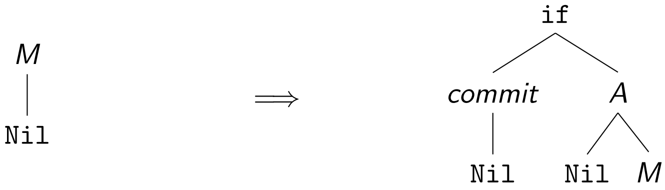

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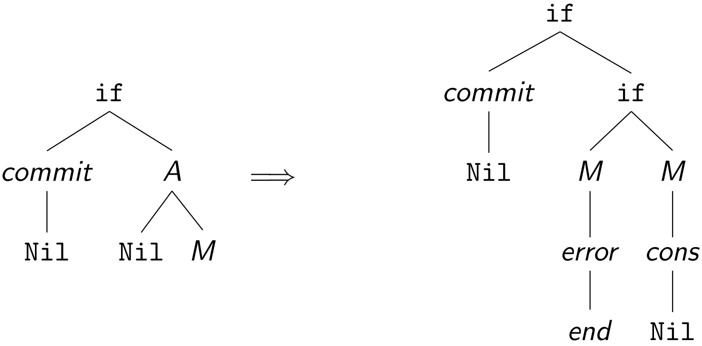

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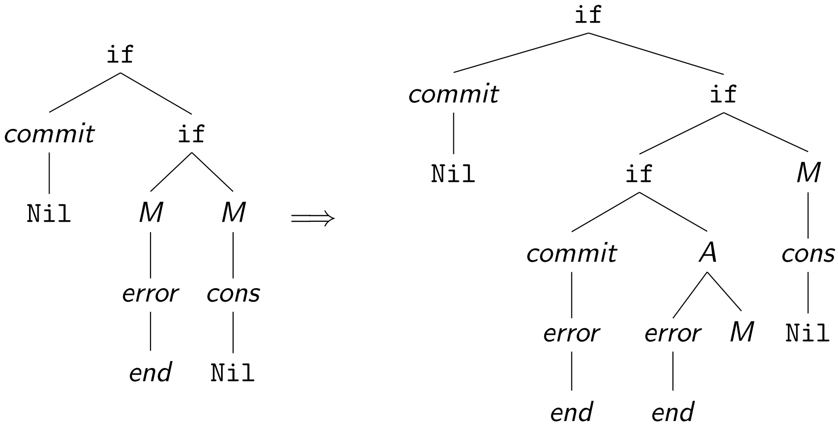

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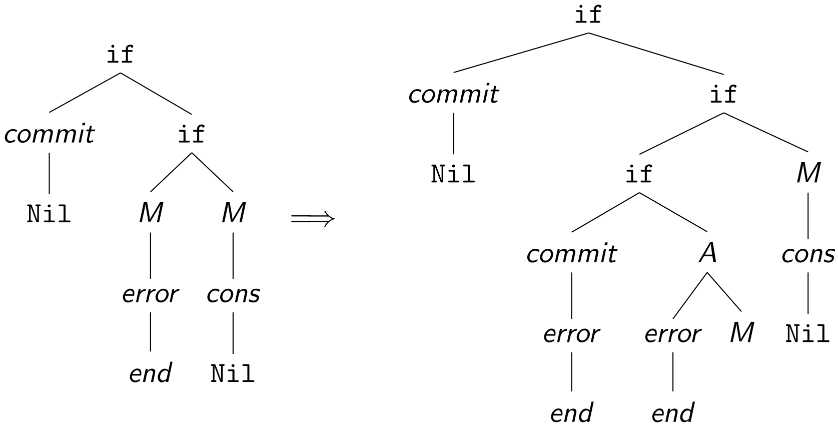

We would like to check that the program modelled by this scheme never commits an error.

Modal μ-calculus

Over trees we may use several logics: CTL, MSO,. . .

In this work we use modal μ-calculus. It is equivalent to MSO over trees.

Grammar: $\phi, \psi ::= X \mid a \mid \phi \vee \psi \mid \phi \wedge \psi \mid \Box\, \phi \mid \diamond_i\, \phi \mid \mu X\,.\,\phi \mid \nu X\,.\,\phi$

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X is a variable

a is a predicate corresponding to a symbol of Σ

$\Box\, \phi$ means that ϕ should hold on every successor of the current node

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We can also define (variant) $\diamond \ = \ \bigvee_i \ \diamond_i$.

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$\mu X.\, \phi$ is the least fixpoint of $\phi(X)$. It is computed by expanding finitely the formula:

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$\nu X.\phi$ is the greatest fixpoint of $\phi(X)$. It is computed by expanding infinitely the formula:

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Specifying a property in modal μ-calculus

How do we specify that the second scheme does not commit an error ?
We want to forbid the existence of an instance of the symbol *error* on a branch after *commit* was seen.

There is an error on a branch $\Longleftrightarrow \mu X.(\ \diamond X \vee \mathit{error}\ )$

There is an error on a branch after a commit
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Over the first example:

- Every branch ends by `Nil`:

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  but is it true ? Take instead

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- There is an infinite branch, the rightmost one, only labelled with `if`.
- Every other branch is finite and ends with a `Nil`.

Interaction with trees: a shift to automata theory

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Alternating parity tree automata

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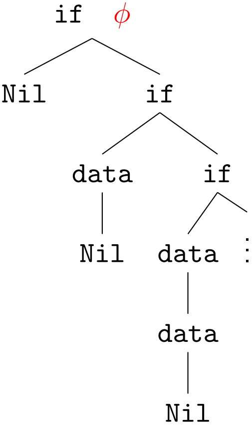

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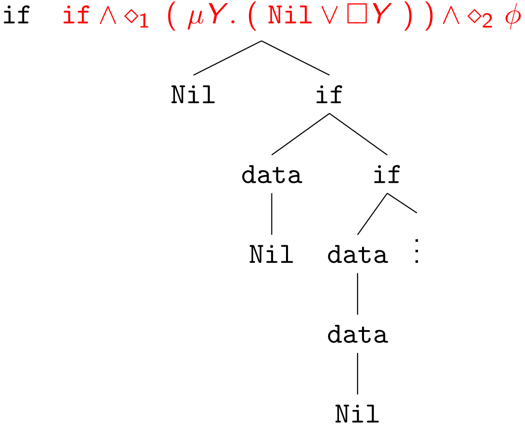

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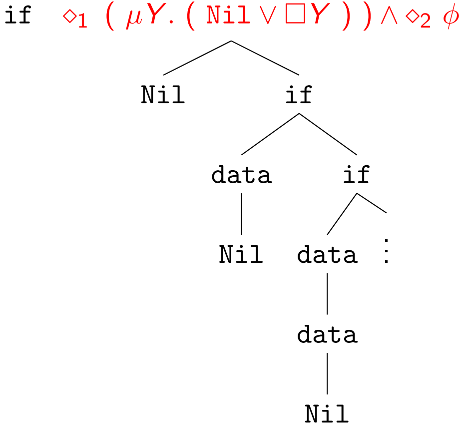

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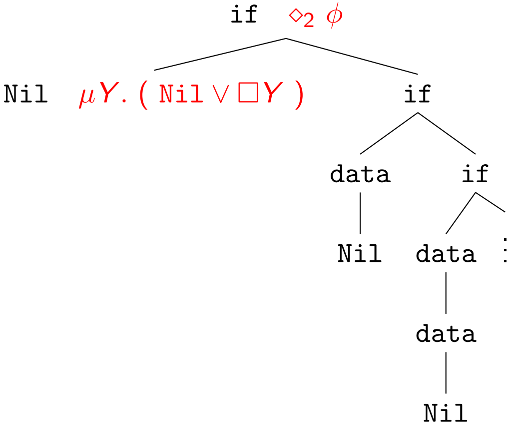

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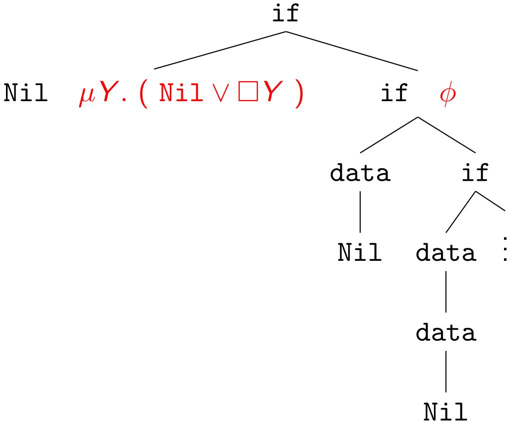

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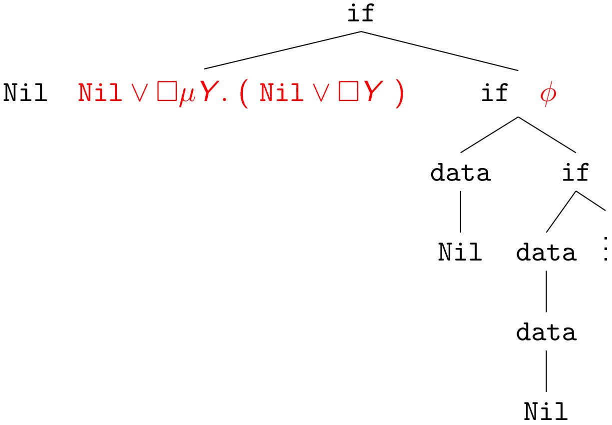

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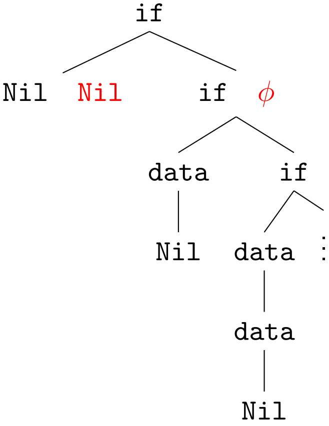

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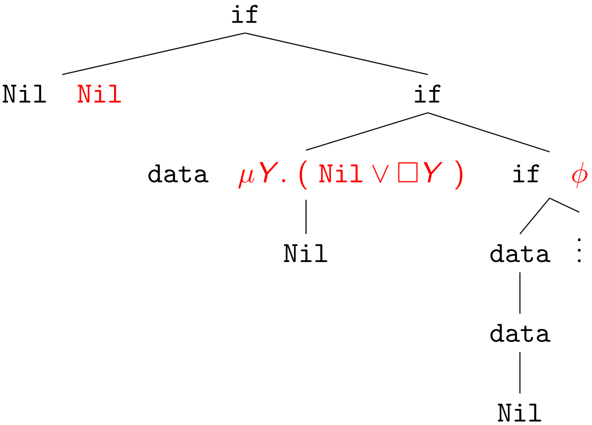

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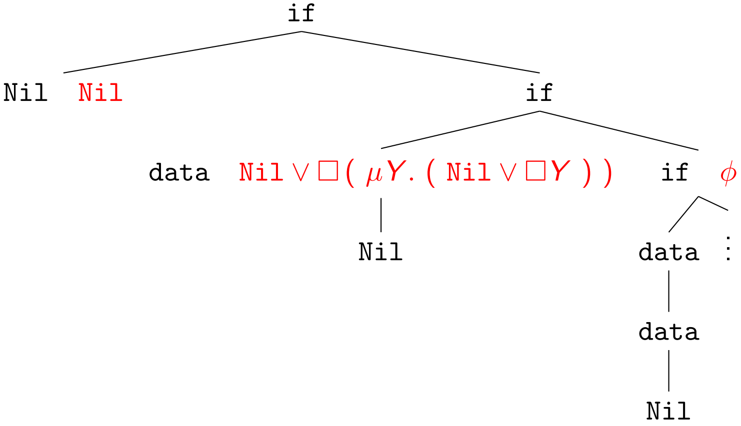

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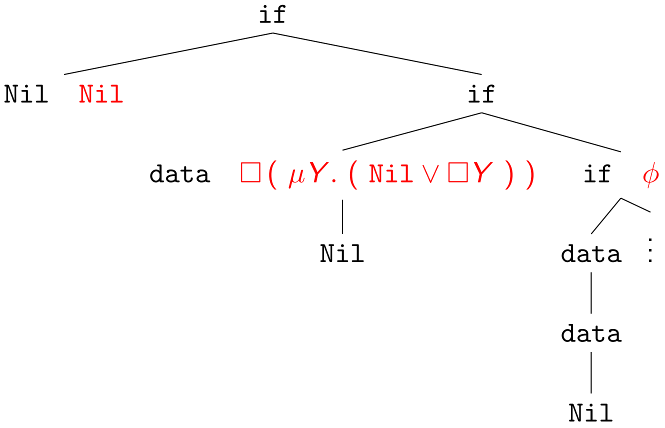

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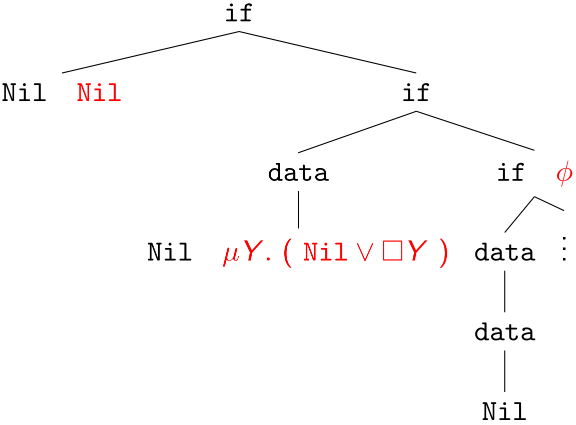

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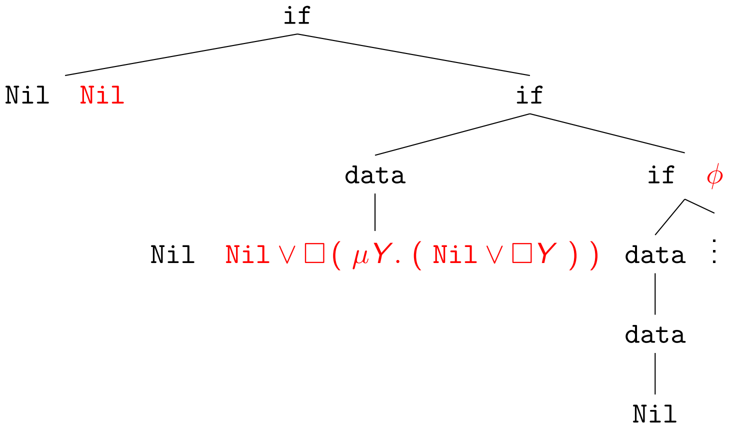

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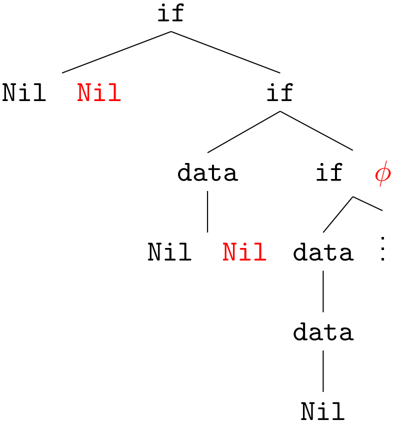

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Conversion to an automaton ?

- Needs to play the formula over the tree, but always by reading a letter.
- Idea: iterate the formula several times until you find a letter.
- Needs non-determinism for $\vee$ and alternation for $\wedge$
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To translate ϕ to an automaton, consider its set of states Q as the set of subformulas of ϕ. Its initial state q_0 corresponds to ϕ, and q_1 to $\mu Y.\,(\;\texttt{Nil} \vee \Box Y\;)$.

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Inductive/coinductive behaviour limitations: you can only play q_1 finitely, but there are no restrictions over q_0.

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In general, transitions may duplicate or drop a subtree.

Example: $\delta(q_0, \texttt{if}) \;=\; (2, q_0) \wedge (2, q_1)$.

Alternating parity tree automata

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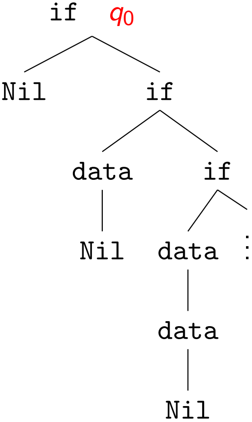

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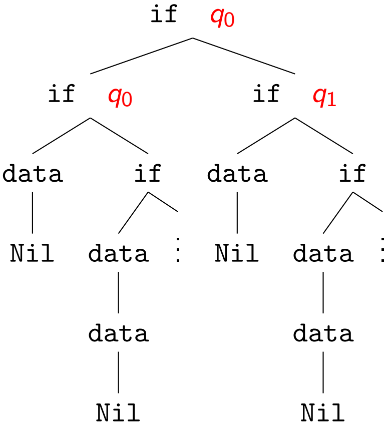

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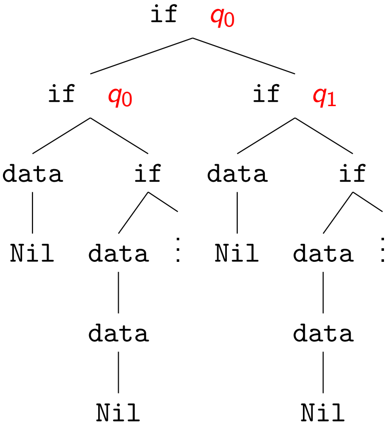

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And for the inductive/coinductive behaviour ?

→ parity conditions

Over a branch of a run-tree, say q_0 has colour 0 and q_1 has colour 1.

Now consider an infinite branch, and the maximal colour you see infinitely often on this branch.

If it is even, accept: it means you looped infinitely on ν.

Else if it is odd the automaton rejects: it means μ was unfolded infinitely, and this is forbidden.

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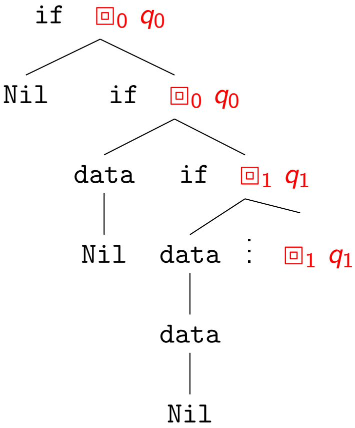
Parity condition on an example



would not be a winning run-tree: the automaton unfolded μ infinitely on the infinite branch (note: δ needs to be modified a little to produce this run-tree).

Alternating parity tree automata

In general, every state is given a colour, and a run-tree is accepting if and only if all its branches have an even maximal infinitely seen colour.

A tree is accepted iff it admits a winning run-tree. This is equivalent to satisfying the modal μ-calculus property encoded by the automaton.

Alternating parity tree automata and intersection types

A key remark (Kobayashi 2009): if $\delta(q, a) \;=\; (1, q_0) \wedge (1, q_1) \wedge (2, q_2) \ldots$

then we may consider that a has a refined intersection type

$$(q_0 \wedge q_1) \Rightarrow q_2 \Rightarrow q$$

and what about colours ?

Consider $(\boxdot_{c_0}\; q_0 \wedge \boxdot_{c_1}\; q_1) \Rightarrow \boxdot_{c_2}\; q_2 \Rightarrow q$

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Alternating parity tree automata and intersection types

This remark is very important, because unlike automata, typing lifts to higher-order.

So we may type a recursion scheme with the states of an automaton to verify if the property it expresses is satisfied.

Very important consequence: remember even silly program models can be not regular. But schemes always are finite — and most of the time rather small.

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A type-system for verification: without colours

$$\text{Axiom}\qquad \frac{}{x\ :\ \bigwedge_{\{i\}}\ \theta_i\ ::\ \kappa\ \vdash\ x\ :\ \theta_i\ ::\ \kappa}$$

$$\delta\qquad \frac{\{\,(i, q_{ij})\mid 1\leq i\leq n, 1\leq j\leq k_i\}\quad \text{satisfies}\quad \delta_A(q,a)}{\emptyset\vdash a\ :\ \bigwedge_{j=1}^{k_1}\ q_{1j}\rightarrow\ \ldots\ \rightarrow\bigwedge_{j=1}^{k_n}\ q_{nj}\rightarrow q\ ::\ \bot\rightarrow\cdots\rightarrow\bot}$$

$$\text{App}\qquad \frac{\Delta\vdash t : (\ \theta_1\ \wedge\cdots\wedge\ \theta_k)\rightarrow\theta\ ::\ \kappa\rightarrow\kappa'\qquad \Delta_i\vdash u\ :\ \theta_i\ ::\ \kappa}{\Delta\ +\ \Delta_1\ +\ \ldots\ +\ \Delta_k\ \vdash\ \ t\,u\ :\ \theta ::\ \kappa'}$$

$$\mathit{fix}\qquad \frac{\Gamma\vdash\mathcal{R}(F)\ :\ \theta\ ::\ \kappa}{F\ :\ \ \theta\ ::\ \kappa\ \vdash\ \ F\ :\ \theta\ ::\ \kappa}$$

$$\lambda\qquad \frac{\Delta\,,\,x\ :\ \bigwedge_{i\in I}\ \ \theta_i\ ::\ \kappa\ \ \vdash\ \ \ t\ :\ \theta\ ::\ \kappa'\qquad\qquad I\ \subseteq\ J}{\Delta\ \ \vdash\ \ \lambda\,x\,.\,t\ :\ \left(\bigwedge_{j\in J}\ \ \theta_j\right)\rightarrow\theta ::\ \kappa\rightarrow\kappa'}$$

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A type-system for verification: example

$$\begin{array}{lcl} \texttt{S} & = & \texttt{L Nil} \\ \texttt{L} & = & \lambda x.\,\texttt{if}\ x\ (\texttt{L}\ (\texttt{data}\ x\ ) \end{array}$$

and transitions:

$$\begin{array}{rcl} \delta(q_0, \texttt{Nil}) = \top & \Leftrightarrow & \texttt{Nil} : q_0 \\ \delta(q_0, \texttt{data}) = (1, q_0) & \Leftrightarrow & \texttt{data} : q_0 \rightarrow q_0 \\ \delta(q_0, \texttt{if}) = (\ (1, q_0) \wedge (2, q_0)\ ) \vee (2, q_1) & \Leftrightarrow & \texttt{if} : (q_0 \rightarrow q_0 \rightarrow q_0) \\ & & \quad \wedge(\emptyset \rightarrow q_1 \rightarrow q_0) \\ \delta(q_1, \texttt{if}) = (2, q_1) & \Leftrightarrow & \texttt{if} : \emptyset \rightarrow q_1 \rightarrow q_1 \end{array}$$

(example on the board — mistakes to check attention only ;-))

A type-system for verification (Grellois-Melliès 2014)

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$$\mathit{fix} \qquad \frac{\Gamma \vdash \mathcal{R}(F) : \theta :: \kappa}{F : \boxdot_{\Omega(\theta)} \theta :: \kappa \ \vdash \ F : \theta :: \kappa}$$

$$\lambda \qquad \frac{\Delta, x : \bigwedge_{i \in I} \boxdot_{m_i} \theta_i :: \kappa \ \vdash \ t : \theta :: \kappa' \qquad I \subseteq J}{\Delta \ \vdash \ \lambda x \,.\, t : \left(\bigwedge_{j \in J} \boxdot_{m_j} \theta_j \right) \to \theta :: \kappa \to \kappa'}$$

A type-system for verification (Grellois-Melliès 2014)

$$\text{Axiom}\qquad \frac{}{x : \bigwedge_{\{i\}} \boxdot_{\Omega(\theta_i)} \theta_i :: \kappa \vdash x : \theta_i :: \kappa}$$

$$\delta \qquad \frac{\{\,(i,q_{ij}) \mid 1 \le i \le n, 1 \le j \le k_i\,\} \text{ satisfies } \delta_A(q,a)}{\emptyset \vdash a : \bigwedge_{j=1}^{k_1} \boxdot_{m_{1j}} q_{1j} \to \ldots \to \bigwedge_{j=1}^{k_n} \boxdot_{m_{nj}} q_{nj} \to q :: \bot \to \cdots \to \bot \to \bot}$$

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A type-system for verification (Grellois-Melliès 2014)

This type system can have infinite-depth derivation.

The parity condition over branches of run-trees may be reformulated as a condition over infinite branches of a derivation tree.

Theorem: there is a winning run-tree over the tree produced by a scheme if and only if there exists a winning derivation of $\vdash S : q_0$ in the type system.

Complexity (Ong): rather huge... n-EXPTIME complete, for n the order of the scheme. But actually not that awfull in practice.

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Consequences and remarks

- We can extend the theorem about the existence of a memoryless strategy for parity games in this setting and give a proof of decidability of model-checking in this way.
- We can work further on the type system and relax some colouring notions. This way we proved that the colouring operation is a modality (a comonad), and interpreted the verification problem in tensorial logic with colouring boxes.
- Many connections with models of linear logic: indexed logic, relational semantics (= run-tree), lattice semantics (= decidability)
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